

IFS Physics Optional Extra Stuff Compilation

YASH DHOBLE
AIR 17 - IFS '22

Contents

① Statistical Physics

- Saha Ionisation Formula
- BEC
- Fermi gas
- Chandrasekhar limit
- Neutron stars & pulsars
- Brownian motion as a random walk
- Negative temperatures

② Quantum Mech: WKB Approximation & use in α decay problem

③ EM: Planck mass / length / time / temp / energy

④ Optics:

- Fresnel Integral
- Cornu Spiral
- diffraction by straight edge
- diffraction by narrow slit

⑤ Mechanics:

- Galilean transformation
- Minkowski Diagrams
- Constraints / DoF / Generalised coordinates
- Lagrangian & Hamiltonian
- Foucault's Pendulum

Sources:

- Std. Books
- Abhijeet Agarwal, IAS Physics Notes
- Youtube & Wiki

iii) Diffusion (Transport of Mass) $\Rightarrow$

Diffusion is a consequence of molecular motion. It occurs in liquids and gases. It is the mixing of different molecules due to molecular motion.

The phenomenon of diffusion is considered as the phenomenon of transport of mass. Diffusion will also take place in a gas if there is a temperature gradient & hence a gradient in random thermal vel. z

Let n be the concn of gas at plane AA' and

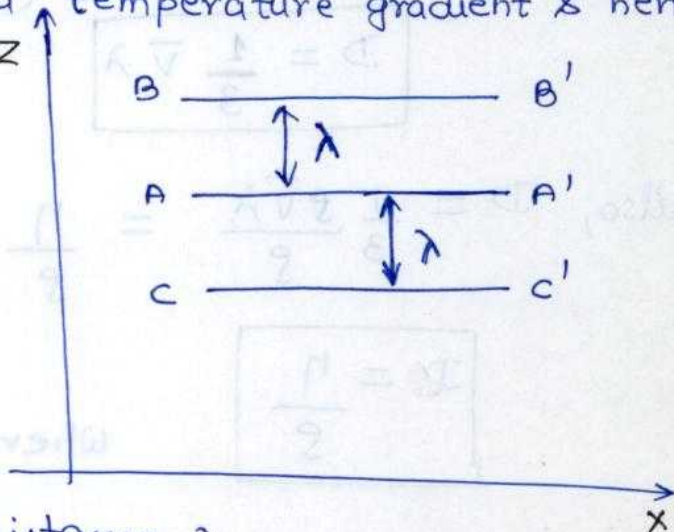
$\frac{dn}{dz}$ the rate of change of concn with distance in an upward direction ($+z$).

Let each of the plane BB' and CC' be at a distance λ

from AA' , where λ is the mean free path.

Then concentration at $BB' = n + \lambda \frac{dn}{dz}$

concentration at $CC' = n - \lambda \frac{dn}{dz}$



The number of molecules from the plane AA' moving downwards per unit area per unit time $= \frac{1}{6} \bar{v} (n + \lambda \frac{dn}{dz})$

The No. of molecules from the plane AA' moving upwards per unit area per unit time $= \frac{1}{6} \bar{v} (n - \lambda \frac{dn}{dz})$

So, Net no. of molecules crossing per unit area of the plane AA' per unit time

$$= \frac{1}{6} \bar{v} (n + \lambda \frac{dn}{dz}) - \frac{1}{6} \bar{v} (n - \lambda \frac{dn}{dz})$$

$$= \frac{1}{3} \bar{v} \lambda \frac{dn}{dz}$$

∴ Coefficient of Diffusion = $\frac{\text{No. of molecules passing per unit area per sec.}}{\text{Concentration gradient}}$

$$D = \frac{\frac{1}{3} \bar{v} \lambda \left(\frac{dn}{dz} \right)}{\left(\frac{dn}{dz} \right)} = \frac{1}{3} \bar{v} \lambda$$

$$\boxed{D = \frac{1}{3} \bar{v} \lambda}$$

also, $D = \frac{1}{3} \frac{\rho \bar{v} \lambda}{\rho} = \frac{\eta}{\rho}$

$$\boxed{D = \frac{\eta}{\rho}}$$

where,

$$\eta = \frac{1}{3} \rho \bar{v} \lambda = \text{Coefficient of viscosity}$$

~~∴~~ $\bar{v} \propto \sqrt{T}$ and $\lambda \propto \frac{T}{p}$

so, $\boxed{D \propto p^{-1} T^{3/2}}$

so, $\boxed{D = \frac{1}{3} \bar{v} \lambda = \frac{\eta}{\rho}}$

Note (Assumptions to be taken)

- i) Considering the diffusion of molecules of a single species into others of same species (self diffusion)
- ii) Assuming the vessel large enough in comparison with mean free path
- iii) Assuming a uniform ~~hydrostatic~~ pressure so that there is no hydrodynamic flow.

Note :-

for the general case of diffusion of one gas through the other, the diffusion coefficient is given by

$$D = \frac{1}{3} \frac{\lambda_1 \bar{v}_1 n_2 + \lambda_2 \bar{v}_2 n_1}{n_1 + n_2}$$

where $\lambda_1, \bar{v}_1$ and n_1 respectively denote the mean free path, avg speed and concⁿ for gas 1 and $\lambda_2, \bar{v}_2$ and n_2 represents the corresponding quantities for gas 2.

cases

i) When $n_2 \gg n_1$ then $D = D_{12} \propto \sqrt{\frac{m_2}{m_1}}$

ii) When $n_1 \gg n_2$, then $D = D_{21} \propto \sqrt{\frac{m_1}{m_2}}$

So, $\frac{D_{21}}{D_{12}} = \left(\frac{m_1}{m_2}\right)$

Question & Answers.

Q.1.) The coefficients of viscosity and diffusion for oxygen are $1.95 \times 10^{-5} \text{ N s m}^{-2}$ and $1.22 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ respectively. If the avg molecular speed is 440 m s^{-1} calculate the density and mean free path of gas molecules.

Ans.

We know that

$$D = \frac{\eta}{\rho}$$

Since $\eta = 1.95 \times 10^{-5} \text{ N s m}^{-2}$, $D = 1.22 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$

$$\therefore \rho = \frac{\eta}{D} = \frac{1.95 \times 10^{-5}}{1.22 \times 10^{-5}}$$

$$\rho = 1.6 \text{ kg m}^{-3}$$

Since, $D = \frac{1}{3} \lambda \bar{v}$

$$\therefore \lambda = \frac{3D}{\bar{v}} = \frac{3 \times 1.22 \times 10^{-5}}{440}$$

$$\lambda = 8.32 \times 10^{-8} \text{ m}$$

Q.2.) Calculate the mass of nitrogen diffusing through an area of 10^{-2} m^2 in 10 sec, if the concⁿ gradient is 1.26 kg m^{-4} , the mean free path is 10^{-7} m and $\bar{v} = 476 \text{ m s}^{-1}$

Ans.

The mass diffusing per unit area per unit time

$$M = \left\{ -\frac{1}{3} \bar{v} \lambda \left(\frac{dn}{dz} \right) \right\} m$$

$$= -\frac{1}{3} \bar{v} \lambda \frac{ds}{dz} \quad (\text{as, } mn = s)$$

$$= \frac{1}{3} \times 476 \times (10^{-7}) \times (1.26)$$

$$M = 2.0 \times 10^{-5} \text{ kg m}^{-2} \text{ s}^{-1}$$

Hence the mass diffusing through an area of 10^{-2} m^2 in 10 sec will be equal to

$$2 \times 10^{-5} \times (10^{-2} \times 10) = 2 \times 10^{-6} \text{ kg}$$

Q.3.) calculate the thermal conductivity of air using the following data: $\rho = 1.29 \text{ kg m}^{-3}$, $\bar{v} = 460 \text{ m s}^{-1}$, $\lambda = 6.4 \times 10^{-8} \text{ m}$, $M = 29$, $\gamma = 1.4$ and $R = 8.31 \text{ kJ kmol}^{-1} \text{ K}^{-1}$

Ans.

Since

$$K = \frac{1}{3} C_v n \bar{v} \lambda$$

$$= \frac{1}{3} \rho \frac{C_v}{M} \bar{v} \lambda$$

Since air is primarily a mix of two diatomic gases (N_2 and O_2), we take $C_v = \frac{5}{2} R$

$$\therefore C_v = 2.5 \times 8.314 \text{ kJ kmol}^{-1} \text{ K}^{-1}$$

$$= 20.8 \times 10^3 \text{ J kmol}^{-1} \text{ K}^{-1}$$

$$\therefore K = \frac{1}{3} \frac{(1.29)(460)(20.8 \times 10^3)(6.4 \times 10^{-8})}{29}$$

$$= 9.07 \times 10^{-3} \text{ J m}^{-1} \text{ s}^{-1} \text{ K}^{-1}$$

$$= 9.1 \times 10^{-3} \text{ W m}^{-1} \text{ K}^{-1}$$

Q.4.) The collision diameters of hydrogen, nitrogen, and chlorine gases are $2.5 \text{ \AA}$, $3.5 \text{ \AA}$ and $4.0 \text{ \AA}$ respectively. Compare their thermal conductivities at a temperature

Ans.

The thermal conductivity is given by

$$K = \frac{1}{3\sqrt{2}} \frac{C_V}{N\pi d^2} \bar{V}$$

Since $\bar{V} = \sqrt{\frac{8kT}{\pi m}}$

$$\therefore K = \frac{1}{3\sqrt{2}} \frac{C_V}{N\pi d^2} \sqrt{\frac{8kT}{\pi m}}$$

At ordinary temperature $C_V = \frac{5}{2}R$ for all the ^{diatomic} ~~atomic~~ gases.

$$\therefore K_{H_2} = \frac{1}{3\sqrt{2}} \frac{C_V}{N\pi d_{H_2}^2} \sqrt{\frac{8kT}{\pi m_{H_2}}}$$

$$K_{N_2} = \frac{1}{3\sqrt{2}} \frac{C_V}{N\pi d_{N_2}^2} \sqrt{\frac{8kT}{\pi m_{N_2}}}$$

$$K_{Cl_2} = \frac{1}{3\sqrt{2}} \frac{C_V}{N\pi d_{Cl_2}^2} \sqrt{\frac{8kT}{\pi m_{Cl_2}}}$$

$$\therefore K_{H_2} : K_{N_2} : K_{Cl_2} :: \frac{1}{d_{H_2}^2 \sqrt{m_{H_2}}} : \frac{1}{d_{N_2}^2 \sqrt{m_{N_2}}} : \frac{1}{d_{Cl_2}^2 \sqrt{m_{Cl_2}}}$$

Since $m_{H_2} = 2/N \text{ gm}$, $m_{N_2} = \frac{28}{N} \text{ gm}$, $m_{Cl_2} = \frac{71}{N} \text{ gm}$

& $d_{H_2} = 2.5 \text{ \AA}$, $d_{N_2} = 3.5 \text{ \AA}$, $d_{Cl_2} = 4.0 \text{ \AA}$

$$\therefore K_{H_2} : K_{N_2} : K_{Cl_2} :: \frac{1}{88} : \frac{1}{65.8} : \frac{1}{134.72}$$

$$K_{H_2} : K_{N_2} : K_{Cl_2} :: 16 : 2 : 1$$



Brownian Motion:-

This phenomenon was first discovered by an English botanist Brown in 1827.

It was observed that if a colloidal solution be examined under an microscope, the suspended particles appears ~~to~~ moving to and fro, rapidly and continuous in an entirely random way.

This irregular motion is called the Brownian motion of the particle.

→ The phenomenon of Brownian motion provides a very useful picture of gaseous state of matter. From the point of view of thermal agitation, the gas molecules are of the same nature as that of the Brownian particles. In fact, all the laws of kinetic theory of gases are applicable to the Brownian particles.

- The observed characteristics of Brownian motion are:-
- i) The motion is continuous, completely random and irregular.
 - ii) No two particles execute the same motion
 - iii) Smaller particles execute faster and hence more noticable motion.
 - iv) The motion ~~is very~~ becomes very ~~vigorous~~ ^{vigorous} and lively when temperature is increased or a less ~~rig~~ viscous liquid is taken.
 - v) The movement is about the same in all directions.
 - vi) The movement (motion) is independent of ~~the~~ external influences.
 - vii) All laws of kinetic theory of gases are applicable to the Brownian particles.

→ Examples of Brownian motion:

- i) sedimentation or pollutants in atmosphere
- ii) Motion of galvanometer mirror.

→ Einstein's theory of the translational Brownian motion:

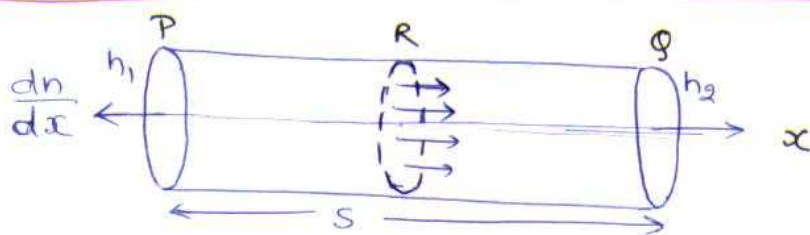
Einstein's theory of translational Brownian motion explains the physical nature of the phenomenon.

Evidently the Brownian particles, due to their random motion, tend to diffuse into the medium in the course of time. Einstein, therefore, related the diffusion coefficient to Brownian motion.

Einstein calculated the diffusion coefficient by considering the irregular motion of suspended particles as well as the ~~os~~ osmotic pressure between different parts.

i) calculation of "D" from random molecular motion:-

For simplicity, we confine ourselves to one dimensional Brownian motion and assume that, on an average, each a particle is displaced through a distance s in time T .



$$\begin{aligned} \bar{c} &= \frac{n_1 A s}{2} \quad \bar{c} = \frac{n_2 A s}{2} \\ T &= \frac{(n_1 - n_2) A s}{F} \\ \frac{T}{A} &= \frac{(n_1 - n_2) s}{2e} = \frac{\delta n}{\delta c} \frac{\delta x}{\delta t} \\ (n_1 - n_2) &= \frac{dn}{dx} \quad \frac{\delta x}{\delta t} = v \Rightarrow D = \frac{s^2}{2T} \end{aligned}$$

Let us imagine a cylinder of cross-sectional area A and length s with its axis parallel to the x -axis. Its end faces are denoted by P and Q . Let the molecular concⁿ at P be n_1 and Q be n_2 such that $n_1 > n_2$. That is, there exists a molecular concentration gradient dn/dx along the cylinder. This urges the suspended particles to diffuse.

Then the no. of particles crossing P to the right in time τ is

$$\vec{I} = \frac{n_1}{2} s A$$

the factor $(1/2)$ arises because only half of the particles contained in a cylinder of volume sA situated left to P will cross it in time τ .

Similarly, the no. of particles entering Q and moving in negative x -direction in time τ is

$$\vec{I} = \frac{n_2}{2} s A$$

So, the excess no. of particles crossing the vertical plane R at the centre of the cylinder in the positive x -direction in time τ is

$$\Gamma = \vec{I} - \vec{I} = (n_1 - n_2) \frac{s A}{2}$$

Hence, the net number of Brownian particles diffusing across R to right per unit area per unit time is

$$\frac{\Gamma}{A \tau} = (n_1 - n_2) \frac{s}{2 \tau}$$

$$\frac{\Gamma}{A \tau} = - \frac{s^2}{2 \tau} \left(\frac{dn}{dx} \right) \quad \text{--- i)}$$

$$\text{Since } (n_1 - n_2) = -s \left(\frac{dn}{dx} \right)$$

But, we know that net number of particles crossing an imaginary surface per unit area per unit time

$$= -D \left(\frac{dn}{dx} \right) \quad \text{--- ii)}$$

where, D is the diffusion coefficient

From eqn i) & ii), we get

$$D = \frac{s^2}{2 \tau}$$

ii) Calculation of D from osmotic pressure difference:

Einstein argued that Van't Hoff's law should hold for dilute solutions as well as dilute suspension.

Let the osmotic pressures exerted by Brownian particles on the faces P and Q be p_1 and p_2 respectively. If we ~~treat~~ treat Brownian particles as the molecules of a gas, we can write

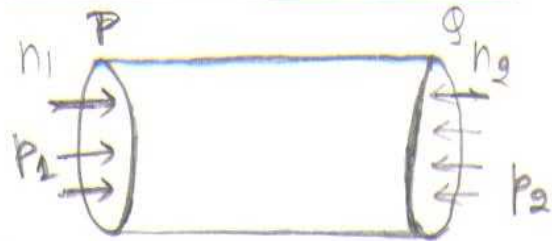
$$p_1 = n_1 k_B T$$

$$\text{and } p_2 = n_2 k_B T$$

where

k_B is Boltzmann Constant

T is temperature of the solution.



Since $(n_1 > n_2)$, p_1 will be greater than p_2 and the osmotic pressure difference $(p_1 - p_2)$ will give rise to a force.

$$F = (p_1 - p_2) A = (n_1 - n_2) k_B T A$$

This force tends to push the cylinder to the right and is experienced by $n A s$ particles contained therein.

Hence, the force experienced by any one of the Brownian particle is

$$f = \frac{F}{n A s} = \frac{(n_1 - n_2) k_B T A}{n A s}$$

$$\text{so, } f = + \frac{k_B T}{n} \frac{(n_1 - n_2)}{s}$$

$$f = - \frac{k_B T}{n} \left(\frac{dn}{dx} \right) \quad \text{--- i) } \left\{ \begin{array}{l} \text{as we know} \\ (n_1 - n_2) = -s \frac{dn}{dx} \end{array} \right.$$

where; n is the mean concⁿ along the cylinder

As a particle moves under the influence of this force, it will experience viscous drag. If we assume that Brownian particles are spherical and have a radius r , then the viscous force is given by Stoke's law and we have

$$f = 6\pi\eta r v = - \frac{k_B T}{n} \left(\frac{dn}{dx} \right)$$

where, η is viscosity of the medium.

or,
$$nv = - \frac{k_B T}{6\pi\eta r} \left(\frac{dn}{dx} \right) \quad \text{--- ii)}$$

The product nv gives the no. of particles moving to the right per second per unit area and is, by definition, equal to $-D \frac{dn}{dx}$ so that

$$D = \frac{k_B T}{6\pi\eta r}$$

comparing the values of D from both eqn's, we get

$$\frac{s^2}{2\tau} = \frac{k_B T}{6\pi\eta r}$$

or,
$$s^2 = \left(\frac{k_B T}{3\pi\eta r} \right) \tau$$

$$s^2 = \left(\frac{RT}{3\pi N_A \eta r} \right) \tau$$

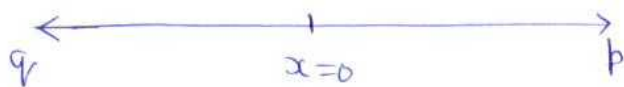
$$\left\{ \text{as } k_B = \frac{R}{N_A} \right.$$

This is known as Einstein's equation, from this it is clear that the observed diffusion of particles is directly related to the molecular properties of gases.

also,
$$\langle x^2 \rangle = \left(\frac{RT}{3\pi N_A \eta r} \right) \tau$$

✓ "Einstein's formula" for mean square displacement of a Brownian particle.

Brownian Motion as a random walk:-



Let us consider a Brownian particle at origin at $t=0$. Suppose that each molecular impact causes it to move some distance. It may or may not change its direction of motion ~~but~~ but the two alternatives are equally likely. That is, it has a probability $(\frac{1}{2})$ of moving in either direction and distributed at random.

Let $W(m, N)$ be the probability that the particle is at the point m after N steps, if it started from the origin. If the particle ~~making~~ makes n_R jumps to the right and n_L to the left, then clearly $n_R + n_L = N$ and

$$n_R - n_L = m$$

By rearranging, these, we can write

$$n_R = \frac{N+m}{2}$$

and, $n_L = \frac{N-m}{2}$

The number of sequences which contain n_R jumps to the right and n_L jumps to the left is equal to the ~~the~~ number of ways of choosing n_L steps to left and n_R steps to the right out of N . Mathematically it is given by $\frac{N!}{n_R! n_L!}$.

The probability of each specified sequence is $(\frac{1}{2})^N$. Hence

✓
$$W(m, N) = \frac{N!}{n_R! n_L!} \left(\frac{1}{2}\right)^N \quad \text{--- i)}$$

$$\text{Or, } W(m, N) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \left(\frac{1}{2}\right)^N \quad \text{--- ii)}$$

Since, N is a very large no., this enables us to reduce eqn (2) to a very convenient form ~~using~~ using sterling's formula.

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + \frac{1}{2} \ln 2\pi$$

Also, using $m \ll N$, eqn (2) reduces to

$$\ln W(m, N) = \frac{1}{2} \ln \left(\frac{2}{\pi N}\right) - \frac{m^2}{2N} \quad \checkmark \text{ msg}$$

$$\therefore \boxed{W(m, N) = \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{m^2}{2N}\right)} \quad \text{--- iii)}$$

Normally, it is more convenient to work with a continuous variable. So if we suppose that the length of each step, say r , is very small, we can write $x = mr$. Then $W(m, N)$ can be approximated by a continuous probability density function

$f(x, N)$ such that

$$W(m, N) = K f(x, N)$$

where, K is a constant.

As the probability of finding the random walker ~~is~~ somewhere is positive definite, K can be calculated by setting

$$\int_{-\infty}^{\infty} f(x, N) dx = 1$$

This gives $K = 2r$

So,

$$f(x, N) = \frac{1}{\sqrt{2\pi N r^2}} \exp\left(-\frac{x^2}{2N r^2}\right) \quad \dots \text{iv)}$$

We note that if the particles take time t in making N jumps and τ denotes the mean free time, then we have

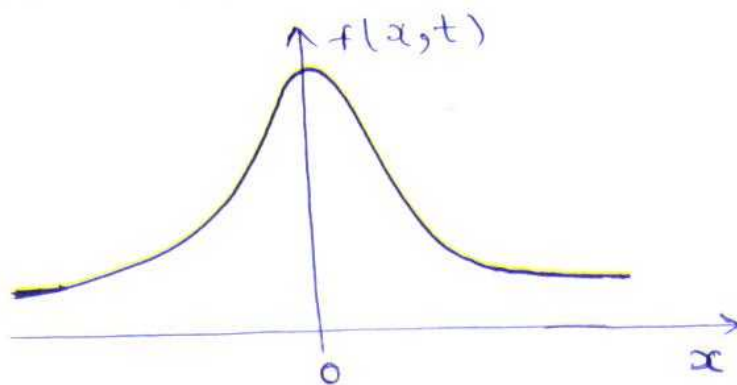
$$N = \frac{t}{\tau}$$

$$\therefore f(x, t) = \frac{\sqrt{\tau}}{\sqrt{2\pi t r^2}} \exp\left(-\frac{x^2}{2 t r^2 \tau}\right)$$

$$\text{or, } f(x, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4 D t}\right)$$

Where, $D = \frac{r^2}{2\tau}$ is known as diffusion coefficient.

This eqn gives the probability that the particle is at a distance x at time t . It represents a Gaussian distribution curve.



Now, the mean square displacement of a Brownian particle is defined as.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x, t) dx$$

$$= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{4 D t}\right) dx$$

Since the integral is even funcⁿ, we can write

$$\begin{aligned}\langle x^2 \rangle &= \frac{1}{2\sqrt{\pi Dt}} \int_0^\infty x^2 \exp\left(-\frac{x^2}{4Dt}\right) dx \\ &= \frac{1}{\sqrt{\pi Dt}} \int_0^\infty x^2 \exp\left(-\frac{x^2}{4Dt}\right) dx.\end{aligned}$$

Now, let us take a new variable, by defining

$$\frac{x^2}{4Dt} = y$$

$$\text{So, } x^2 dx = 4(Dt)^{3/2} y^{1/2} dy$$

The substituting, we get

$$\begin{aligned}\langle x^2 \rangle &= \frac{4Dt}{\sqrt{\pi}} \int_0^\infty y^{1/2} \exp(-y) dy \\ &= \frac{4Dt}{\sqrt{\pi}} \Gamma(3/2)\end{aligned}$$

Where $\Gamma(x)$ is gamma funcⁿ & $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ so we get

$$\boxed{\langle x^2 \rangle = 2Dt} \quad \checkmark$$

$$\text{Since } D = \frac{k_B T}{6\pi\eta r}$$

so, using this, we get

$$\langle x^2 \rangle = \frac{k_B T t}{3\pi\eta r}$$

$$\therefore \frac{1}{k_B} = \frac{T t}{3\pi\eta r \langle x^2 \rangle}$$

$$\left\{ \text{Since } k_B = \frac{R}{N_A} \right.$$

$$\therefore \boxed{\frac{N_A}{R} = \frac{T t}{3\pi\eta r \langle x^2 \rangle}}$$

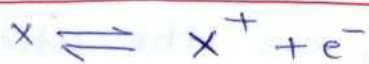
By measuring Avogadro no., we can vindicate the kinetic theory of gases.

Saha Ionisation Formula.

A positive ion is produced by the removal of one or more electrons from a neutral atom or molecule. This process may be caused by thermal agitation, and is called thermal ionisation.

- i) The ionisation begins at a temperature of the order of 6000°C .
- ii) The average kinetic energy of molecule $(=\frac{3}{2}kT)$ is then sufficient to cause frequent collisions among the molecules which supply the energy for ionisation. The degree of ionisation depends on the temperature.

Saha considered the thermal ionisation process to be similar to a reversible chemical reaction



Where x^{+} is the positive x ion and e^{-} the electron. On the direct process, x is ionised and in the reversible process, recombination takes place. The two processes take simultaneously till there is equilibrium, when the concentration of ~~x^{+}~~ x^{+} and e^{-} become constant.

For a given temperature, the total pressure is fixed. In the ionised gas, the electrons are spin $\frac{1}{2}$ particles. The neutral x atoms are bosons. ^{But} the ionised x ion has no spin and is a boson. Thus the ionised gas is a mixture of fermions and bosons.

Saha obtained the equation of state for such a gas, Saha's ionisation formula, which relates the temperature, pressure and ionisation

potential of atoms to their degree of ionisation.

- To arrive at Saha's formula, we assume that
- a) the contribution of nuclear spin is negligible
 - b) only single ionisation occurs
 - c) dynamic equilibrium beth ionisation and recombination is reached at a given temperature and pressure, so that



where x^+ is ~~singl~~ singly positive ion.

$$\text{since } dG = v dP - s dT + \sum \mu_j dN_j$$

so at equilibrium (at constant T & P), $dG = 0$

$$\text{So, } \sum \mu_j dN_j = 0$$

$$\text{so, } \mu_a - \mu_i - \mu_e = 0$$

$$(\text{as } dN_e = dN_a = dN_i)$$

where μ_a , μ_i , and μ_e are the chemical potential of atom x , ionised atom x^+ and electron respectively.

Now, expressing μ as a funch of T, p etc.

$$\mu_j = -k_B T \ln \left[\left(\frac{2\pi m_j}{h^2} \right)^{3/2} \frac{\bar{z}_j(T)}{p_j} \frac{k_B T}{p_j} \right]$$

where

$\bar{z}$ is the partition funch

p is pressure.

~~sim~~ so putting expression of μ_j for atom, ionised atom and electrons, we get Saha's ionisation formula as.

$$\log_{10} \left(\frac{x^2 P}{1-x} \right) = - \frac{U}{2.3 RT} + \frac{5}{2} \log_{10} T - 6.459$$

Where, x is fraction of atoms which are ionised

P : total pressure

U : ionisation energy, T : temperature.

Saha's eqnⁿ shows that the degree of ionisation α depends not only on temperature but also on pressure. It is large at high temperatures and Low pressure.

This formula finds many applications in astrophysics.^{Some of} which are

i) Fraunhofer spectrum of the Sun:

At the lower levels of the sun's atmosphere both the temperature and pressure are high. The high pressure masks the effect of high temperatures, at a lower layers. Therefore when we examine the light from the lower levels of the sun's atmosphere, there is an abundance of the lines of unionised atoms.

At higher levels of Sun's atmosphere, there are lines of ionised atoms instead of unionised atoms.

ii) It is used to estimate the temperature of the atmosphere. add from B.T. solⁿ 2001

iii) Now a days, it is finding wide applications in plasma studies, as also in the ionisation processes in semi-conductors.

Bose Einstein Condensation:

There is no restriction on the number of bosons which can be in any single particle ~~stage~~^{state}. The 'collapse' of bosons into a single particle ground state is called the "Bose-Einstein Condensation". It is not a condensation in space but can be called a condensation in momentum space.

Let us consider of a gas of N bosons in a box of volume V . If N_g be in the ground state and N_{ex} in the excited state then

$$N = N_g + N_{ex}.$$

The mean ~~no.~~^{fraction} of particle in i th state is

$$f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} - 1} = \frac{1}{e^{(\epsilon - \mu)/k_B T} - 1} \quad \text{--- i)}$$

So, the no. of particles lying beth energy range ϵ and $\epsilon + d\epsilon$ is given by

$$dn(\epsilon) = f_{BE}(\epsilon) \cdot g(\epsilon) d\epsilon$$

and

$$g(\epsilon) = g_b \frac{4\pi m V}{h^3} (2m\epsilon)^{1/2} = C \epsilon^{1/2}$$

where, $g_b = (2s+1)$ is multiplicity caused by spin of boson.

$$\therefore, C = \frac{4\pi V g_b}{h^3} (2m)^{3/2}$$

$$\therefore dn(\epsilon) = \frac{C \epsilon^{1/2} d\epsilon}{e^{(\epsilon - \mu)/k_B T} - 1} = \frac{C \epsilon^{1/2} d\epsilon}{\frac{1}{A} e^{\epsilon/k_B T} - 1}$$

$$dn(\epsilon) = \frac{C \epsilon^{1/2} d\epsilon}{\frac{1}{A} e^{\epsilon/k_B T} - 1} \quad \text{where } A = e^{\mu/k_B T} \quad \text{--- ii)}$$

$$\text{So, } N = \int_0^{\infty} \frac{c \epsilon^{1/2}}{\frac{1}{A} e^{\epsilon/k_B T} - 1} d\epsilon \quad \text{--- iii)}$$

If we choose the ground state energy as the zero of our energy scale, then for the ground state, $\epsilon = 0$ and so $g(\epsilon) = 0$. So the ground state is not included in the above integration, which now gives N_{ex} , the no. of bosons in the excited state

$$\text{So, } N_{ex} = \int_0^{\infty} \frac{c \epsilon^{1/2} d\epsilon}{\frac{1}{A} e^{\epsilon/k_B T} - 1}$$

For ground state, $\epsilon = 0$ so the no. of bosons in the ground state is given by

$$N_g = \frac{1}{\frac{1}{A} - 1} \quad \left\{ \begin{array}{l} \text{putting } \epsilon = 0 \text{ in eqn i)} \\ \& A = e^{\epsilon/k_B T} \end{array} \right.$$

~~So~~, Therefore the total no. n of bosons is

$$N = N_g + N_{ex}$$

$$N = N_g + \int_0^{\infty} \frac{c \epsilon^{1/2} d\epsilon}{\frac{1}{A} e^{\epsilon/k_B T} - 1} \quad \text{--- iv)}$$

$$\text{Now, } N_{ex} = \int_0^{\infty} \frac{c \epsilon^{1/2} d\epsilon}{\frac{1}{A} e^{\epsilon/k_B T} - 1}$$

To evaluate this integral, we take the substitution $\epsilon/k_B T = x$ then.

$$\therefore \epsilon = k_B T \cdot x \Rightarrow d\epsilon = k_B T dx.$$

$$\epsilon^{1/2} d\epsilon = (k_B T)^{3/2} x^{1/2} dx$$

$$\therefore N_{ex} = C (k_B T)^{3/2} \int_0^{\infty} \frac{x^{1/2} dx}{e^x - 1}$$

(assuming $A=1$ for completely degenerate gas)

$$\therefore \boxed{N_{ex} = C (k_B T)^{3/2} \Gamma(3/2) \zeta(3/2)} \quad \text{--- v.}$$

Where,

$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2} \quad (\text{gamma function})$$

$$\& \quad \zeta(3/2) = 2.612 \quad (\text{Riemann-Zeta function of order } 3/2)$$

$eqn^h(v)$ shows that no. density of excited particles is a function of temperature. As $T \rightarrow 0$, $N_{ex} \rightarrow 0$ and $N_g \rightarrow N$ i.e. all particles condense into ground state. This phenomenon is referred to as "Bose-Einstein Condensation".

However as T increases N_{ex} also increases, it may become arbitrary large. But N is finite and N_{ex} has to be necessarily less than or equal N . We therefore postulate that $eqn^h(v)$ holds only as long as $N_{ex} < N$. If T_b is the maximum temperature which satisfies it, then

$$N_{ex} = C \zeta(3/2) \Gamma(3/2) (k_B T)^{3/2} \quad \text{for } T \leq T_b$$

$$= N \quad \text{for } T > T_b$$

That is, at low temperature, the no. of atoms in excited state increases as $T^{3/2}$ until all atoms are in the excited state at temp T_b . so we can write.

$N_{ex} = N$ at $T = T_b$, we have

$$N = C \Gamma(3/2) \zeta(3/2) (k_B T_b)^{3/2} \quad \text{--- vi)}$$

$$\text{or, } T_b = \frac{1}{k_B} \left[\frac{N}{C \Gamma(3/2) \zeta(3/2)} \right]^{2/3}$$

$$= \frac{h^2}{2\pi m k_B} \left[\frac{N}{2.612 g_b V} \right]^{2/3}$$

$$\left\{ \because C = \frac{2\pi V g_b (2m)^{3/2}}{h^3} \right.$$

$$\therefore T_b = \frac{h^2}{2m\pi k_B} \left[\frac{N}{2.612 g_b V} \right]^{2/3} \quad \text{--- vii)}$$

So, T_b is known as the "Bose-Einstein Condensation Temperature", which is equal to that temperature at which all the bosons should be in the excited states.

- T_b depends upon the particle mass and particle density (N/V) in the system.
- Above T_b , the no. of particles (bosons) in the ground state is almost negligible. But below T_b , particles begin to accumulate in the quantum state with $\epsilon=0$. Accordingly, $T < T_b$ a boson gas may be looked upon as a mixture of two phases. i) a normal phase, consisting of N_{ex} particles distributed over the excited states ($\epsilon \neq 0$) and a condensed phase, consisting of N_g particles accumulated in the ground state ($\epsilon=0$). NOW, we can write N_g and N_{ex} in terms of N , T and T_b .

Hence from eqn vi) & vii), we have

$$\frac{N_{ex}}{N} = \left(\frac{T}{T_b} \right)^{3/2}$$

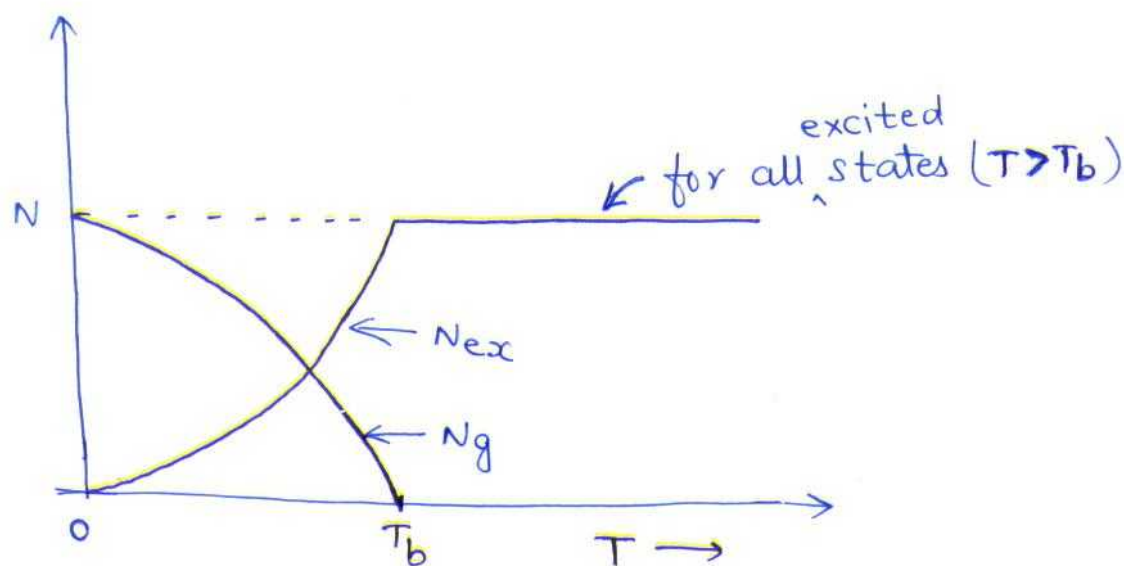
$$\text{or, } N_{ex} = N \left(\frac{T}{T_b} \right)^{3/2} \quad \text{--- viii)}$$

and,
$$N_g = N - N_{ex} = N \left[1 - \frac{N_{ex}}{N} \right]$$

$$N_g = N \left[1 - \left(\frac{T}{T_b} \right)^{3/2} \right] \quad \text{--- ix}$$

It shows that at $T=0$, all particles condense into the lowest energy state ($\epsilon=0$) or zero momentum state.

The figure below shows how N_{ex} and N_g vary with temperature.



If we use $\frac{N}{V} = 2.2 \times 10^{28} \text{ m}^{-3}$ and $m = 6.65 \times 10^{-27} \text{ kg}$

and $\text{spin} = 0$ in the expression for T_b

we get $T_b = 3.13 \text{ K}$.

which is close to the observed value of 2.18 K for the onset of condensation in liquid helium.

Thus the 2.18 K phase transition from He-I to He-II can be explained by Bose-Einstein Condensation.

It tells us that condensation into ground state is a necessary condition for the occurrence of superfluid behaviour.

Helium and Bose-Einstein Condensation

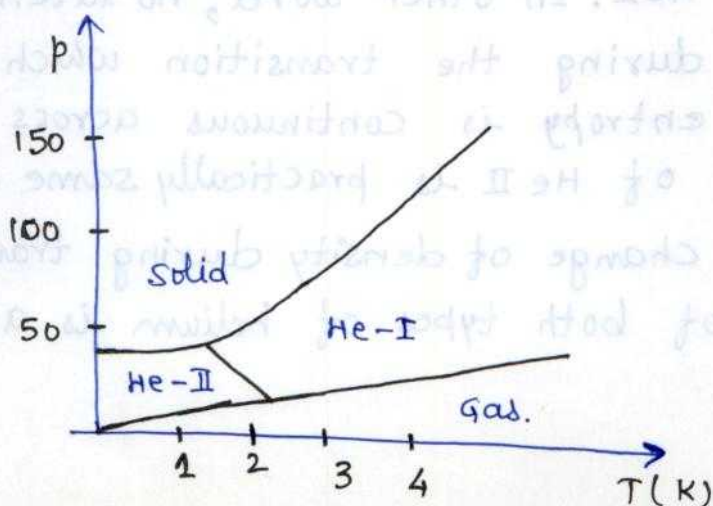
At atmospheric pressure, helium condenses into a normal liquid at 4.2 K. As the temperature is increased, lowered further, liquid helium exhibits another phase transition at 2.18 K. You may expect helium to solidify instead it changes into another liquid of very surprising, in fact unique properties. The new phase is called "liquid helium-II" to distinguish it from the phase above 2.18 K, which is termed "liquid He-I".

Unique properties of "liquid He-II"

- i) It has practically zero viscosity and can flow rapidly almost without resistance, through narrow tubes. The property associated with it have been characterised as super fluidity.
- ii) High heat Conductivity: It has infinite thermal conductivity. This means that irrespective of where you heat the liquid, it will evaporate from the top surface. The He-I-He-II phase transition is visually characterised by the disappearing of bubbles and boiling.

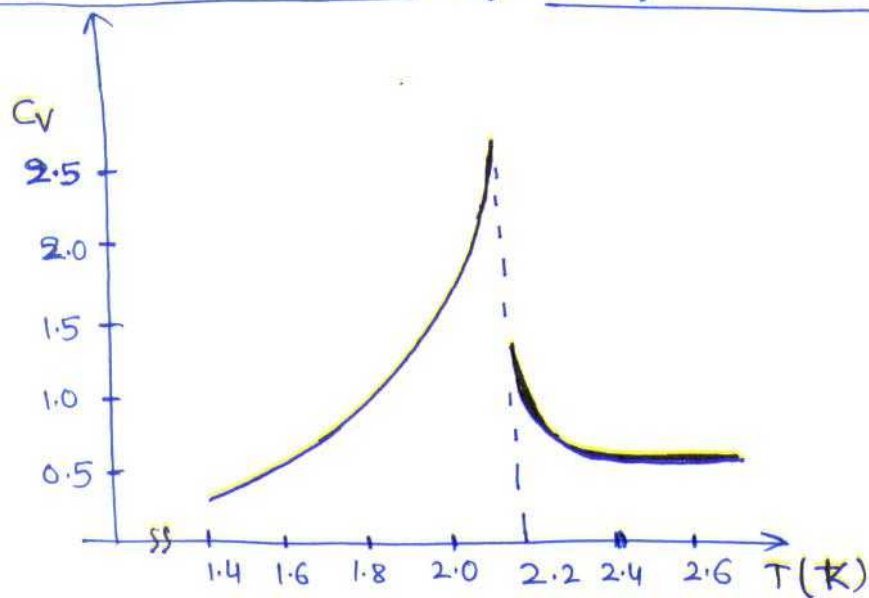
Other features.

a.) p-T diagram of ^4He .

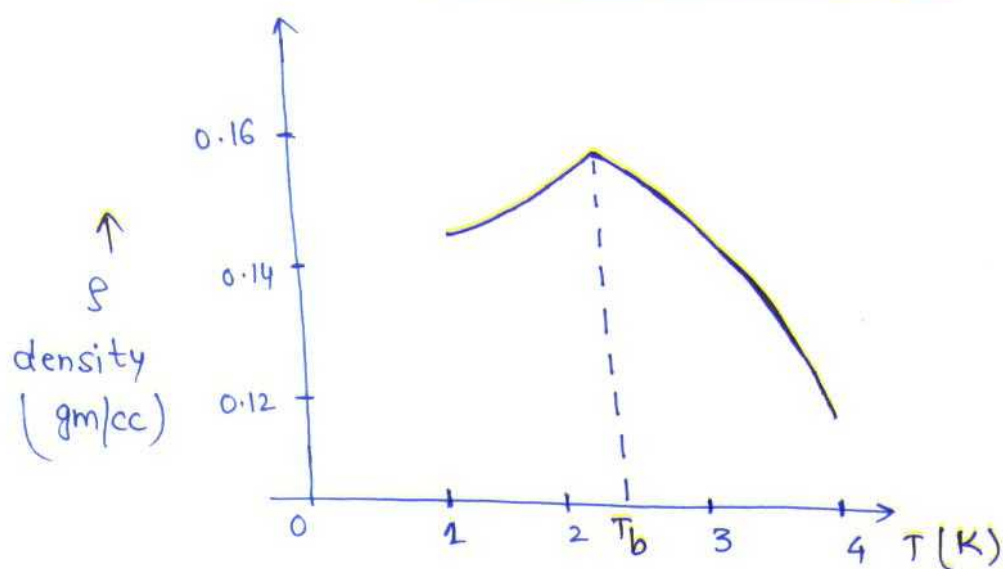


helium has no triple point.

b.) Temperature variation of C_v of He near λ point



c.) Density variation with temperature:



Conclusion:

- i) No heat is involved or absorbed during the transition from He I $\rightarrow$ He II. In other word, no latent heat is involved during the transition which suggest that the entropy is continuous across the curve.
i.e. Entropy of He II is practically same as that of He-I.
- ii) There is no change of density during transitions
i.e. density of both types of helium is about the same.

Chandrasekhar Limit:

The core of a star that has exhausted its nuclear fuel can end up in different ways depending on its mass. S. Chandrasekhar discovered theoretically in 1930s that a stable condition of equilibrium between the collapsing gravitational force and the degeneracy pressure of Fermi gas (electron) can occur with cores which have masses up to " $1.4 M_0$ " where M_0 is the mass of the sun. This is known as the "Chandrasekhar Limit". Such stable cores of dead stars are called "White Dwarfs". If the mass of the core exceeds the Chandrasekhar limit the gravitational compression becomes so large that the electrons are forced into the nuclei. This process is known as "inverse beta decay". The star is now only composed of neutrons. These neutrons can withstand the gravitational compression. We have a "Neutron Star."

Derivation:

A small star consisting of elements lighter than iron which has reached the stage where no further nuclear burning is possible is called a white dwarf. Despite the high temperature found in stellar interiors, the electrons in a white dwarf are quite degenerate because the value of Fermi Temperature T_F is so large. When a star burns all its hydrogen fuel it is made up mainly of plasma of helium nuclei and electrons. The gravitational forces between the helium nuclei cause the star to collapse until the Pauli pressure of the electrons given by

$$P = \frac{2}{5} \rho \epsilon_F(0) \quad \text{--- i)}$$

brings a halt to the collapse.

Here,

ρ = density of the gas
 $\epsilon_F(0)$ = fermi energy at absolute zero.

$$\begin{aligned} n_F &= \frac{N}{V} \Rightarrow \begin{cases} pV = \frac{2}{5} E \\ p = \frac{2}{5} n \epsilon_F \end{cases} \end{aligned}$$

The negative (inward) gravitational pressure is

$$P_G \sim \frac{E_G}{V} \sim \frac{GM^2}{R} \frac{1}{R^3} \quad \text{--- -- -- ii)}$$

Here, E_G is the gravitational potential energy

G : Gravitational Constant

R : Radius of the star

$M \sim N m_{He}$ is the mass of ^{the} star made up of N helium atoms each of mass m_{He} .

The outward Pauli pressure is given by

$$P_p \sim \frac{2}{5} \frac{N}{V} \cdot \frac{h^2}{2m_e} \left(\frac{3N}{8\pi V} \right)^{2/3} \quad \left[\because E_F = \frac{h^2}{2m_e} \left(\frac{3N}{8\pi V} \right)^{2/3} \right]$$

$$\sim \frac{N}{R^3} \frac{h^2}{m_e} \left(\frac{N}{R^3} \right)^{2/3} \quad \left[\because V \sim R^3 \right] \quad \text{--- -- -- iii)}$$

Here m_e is the electron mass, and the no. of electrons is actually $2N$, twice the no. of helium nuclei. Equating P_p and P_G gives the radius of the star.

$$\therefore \frac{G N^2 m_{He}^2}{R^4} \approx \frac{N^{5/3} h^2}{m_e R^5}$$

$$\Rightarrow R \approx \frac{N^{5/3} h^2}{G m_e m_{He}^2 N^2} \approx \frac{h^2}{G m_e m_{He}^2 N^{1/3}}$$

$$\therefore \boxed{R \sim \frac{h^2}{G m_e m_{He}^{5/3} M^{1/3}}} \quad (\because M = N m_{He}) \quad \text{--- -- -- iv)}$$

All the numerical constants from this expression have been dropped to get just an ~~another~~ order of magnitude estimate, putting $M = M_0$ (the solar mass),

We get $R = 7 \times 10^2$ Km, which shows that white dwarf is a compact object.

Equⁿ (iv) is derived on the basis of non-relativistic theory. But the electron system of the white dwarf must be treated relativistically. The relativistic treatment yields the result that for stellar masses greater than about $1.4 M_{\odot}$, the Pauli Pressure can no longer stop the gravitational collapse. The star continues its infall, a supernova occurs, and ultimately a neutron star or a black hole is formed. This critical mass is known as "Chandrasekhar limit".

Neutron Stars and Pulsars.

Larger stars have sufficient helium for the nuclear reaction to continue so that heavier elements up to iron are formed. But iron is the heaviest element that can be formed with the production of energy. As iron fills up the core of the massive star, the temperatures are so high that the iron nuclei begin to break into smaller units like α -particles. The pressure is no longer high enough to counteract gravity and the core collapses. As the density increases, the electrons are squeezed into the nuclei and react with the protons, there to produce neutrons and ~~anti~~neutrinos.



The neutrino escape. A gas composed mainly of neutrons is left behind in the dense core as the outer layers explode as a supernova. Now the core has a very high density. At this density a condition called neutron degeneracy, in which the neutrons cannot be packed any more tightly, appears. The pressure caused by neutron degeneracy balances the gravitational force that tends to collapse the core. As a result the core

reaches equilibrium as a neutron star.

Typically neutron stars have radius of about 10 km although in mass they are comparable of sun. ~~they~~ The chunk of their matter, the size of a small sugar cube contains about 10^8 tons of neutrons. Neutron stars have mass between 1.4 and $\sim 3 \times$ ^{time} Mass of Sun.

Before it collapses, the core has only a weak magnetic field. But as the core collapses, the magnetic field is concentrated, and grows stronger as a result. By the time the core ~~shrinks~~ to neutron star size, it has an extremely powerful magnetic field.

Pulsars: First discovered in 1968, a pulsar is believed to be a rotating neutron star. They emit radiation from a relatively small spot on the star. This situation produces an effect rather like the beam of a light house. The distant observer receives a pulse of radiation as the beam sweeps across ~~its~~ his position on each rotation of the star.

A pulsar gives off pulses of radio waves. A very few objects (Crab and Vela pulsars) are known pulse γ -rays, x-rays or visible light. The periods of known pulsars range between 33 milliseconds and 3.75 seconds. Pulse durations range from 2 to about 150 milliseconds with longer period pulsars generally having a longer pulse duration.

In 1974 Hulse and Taylor discovered what is called a binary pulsar which consists of a pulsar and companion star (believed to be another neutron star) rotating like a dumbbell. Their orbit is about the size of the Sun and the period is about 8 hours. Like a rotating dumbbell the orbiting binary system should emit gravitational radiation.

$$\sum_i \left[\log \left(\frac{g_i - n_i}{n_i} \right) + \alpha - \beta \epsilon_i \right] dn_i = 0$$

as the variations dn_i are independent of each other, we get

$$\log \left(\frac{g_i - n_i}{n_i} \right) + \alpha - \beta \epsilon_i = 0$$

$$\text{or, } \frac{g_i - n_i}{n_i} - 1 = e^{-\alpha + \beta \epsilon_i}$$

$$\text{or, } \frac{g_i}{n_i} = \frac{e^{-\alpha + \beta \epsilon_i}}{+1}$$

$$\text{or, } n_i = \frac{g_i}{e^{-\alpha + \beta \epsilon_i} + 1}$$

$$n_i = \frac{g_i}{e^{-\alpha} e^{\beta \epsilon_i} + 1}$$

This equation represents the most probable distribution of the particles among various energy levels for a system obeying fermi-Dirac statistics and is therefore known as fermi-Dirac distribution law.

fermi-Dirac Gas.

→ An assembly of fermions (i.e. indistinguishable particles of half integral spin obeying Pauli exclusion principle) is known as fermi-Dirac gas.

According to fermi-Dirac distribution law, the most probable distribution is given by

$$n_i = \frac{g_i}{e^{-\alpha + \beta \epsilon_i} + 1}$$

where $\alpha = \frac{\mu}{kT}$ and $\beta = \frac{1}{kT}$

Thus, it may be written as

$$n_i = \frac{g_i}{e^{(\epsilon_i - \mu)/kT} + 1}$$

While dealing with a continuous system we replace n_i by $n(\epsilon)$ and g_i by $g(\epsilon)$ because energy levels may be spaced so closely that their discrete nature is not observable. So

$$n(\epsilon) = \frac{g(\epsilon)}{e^{(\epsilon - \mu)/kT} + 1}$$

We introduce the Fermi-distribution function $f(\epsilon)$ defined by

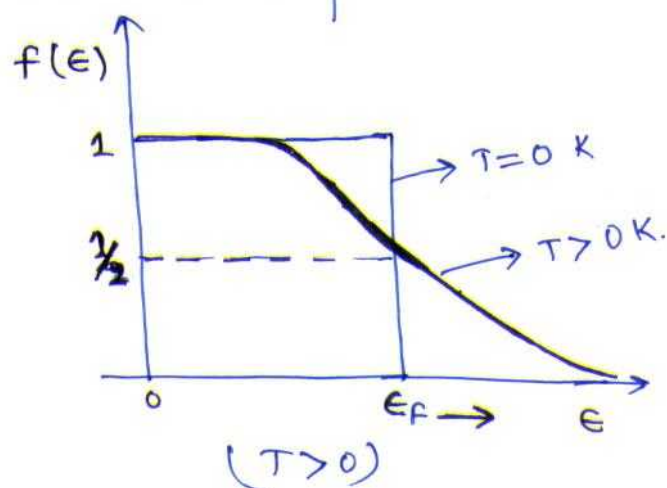
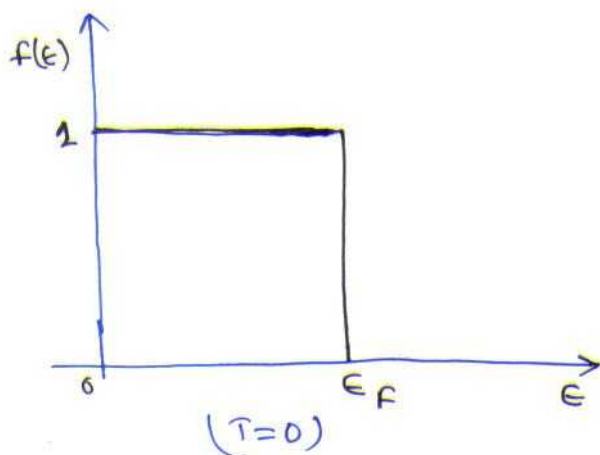
$$f(\epsilon) = \frac{n(\epsilon)}{g(\epsilon)} = \frac{1}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

$$f(\epsilon) = \frac{1}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

at $T = 0 \text{ K}$,
 $f(\epsilon) = 1$
 $= 0$

for $\epsilon < \epsilon_F$
 for $\epsilon > \epsilon_F$

at any Temp T
 $f(\epsilon) = \frac{1}{2}$ for $\epsilon = \epsilon_F$



Fermi Energy: The value of chemical potential at $T = 0 \text{ K}$ corresponding to the energy of the highest occupied energy level is called the Fermi energy, (ϵ_F) .

→ Derivation of ϵ_f :

Consider a system of N fermions enclosed in volume V .

The the no. of quantum states of a particle with momentum, p and $p+dp$ is

$$g(p) dp = g_s \frac{4\pi p^2 V dp}{h^3}$$

✓ where, $g_s =$ degeneracy factor caused by the particle spin s

$$g_s = (2s+1)$$

Since $\epsilon = \frac{p^2}{2m}$ or, $dp = \left(\frac{m}{2\epsilon}\right)^{1/2} d\epsilon$

then the no. of states between energy ϵ and $\epsilon+d\epsilon$ is given by

$$g(\epsilon) d\epsilon = g_s \cdot \frac{4\pi V m \sqrt{2m\epsilon}}{h^3} d\epsilon \quad \text{--- i)}$$

Thus ~~total~~ the no. of fermions with energy between ϵ and $\epsilon+d\epsilon$ is

$$dn(\epsilon) = f(\epsilon) g(\epsilon) d\epsilon = g_s \frac{4\pi m V}{h^3} \sqrt{2m\epsilon} f(\epsilon) d\epsilon$$

Therefore the total no. of particles in Fermi-Dirac gas is given by

$$N = \int_0^{\infty} dn(\epsilon) = \int_0^{\infty} g_s \frac{4\pi m V}{h^3} \sqrt{2m\epsilon} f(\epsilon) d\epsilon.$$

at $T=0K$,

$$f(\epsilon) = 1 \quad \text{for } \epsilon \leq \epsilon_f$$

$$= 0 \quad \text{for } \epsilon > \epsilon_f.$$

$$N = \int_0^{\epsilon_f} g_s \frac{4\pi m V}{h^3} \sqrt{2m\epsilon} d\epsilon + \int_{\epsilon_f}^{\infty} g_s \frac{4\pi m V}{h^3} \sqrt{2m\epsilon} d\epsilon \cdot (0)$$

$$\therefore N = \int_0^{\epsilon_F} g_s \frac{4\pi m V}{h^3} \sqrt{2m} \epsilon^{1/2} d\epsilon$$

$$N = g_s \frac{4\pi V}{h^3} \frac{(2m)^{3/2}}{3} \epsilon_F^{3/2} = \frac{4\pi V}{3h^3} g_s (2m)^{3/2} \epsilon_F^{3/2}$$

$$\therefore \epsilon_F = \frac{h^2}{2m} \left(\frac{3N}{4\pi V g_s} \right)^{2/3}$$

$$\epsilon_F = \frac{h^2}{2m} \left(\frac{3n}{4\pi g_s} \right)^{2/3}$$

where,

$$n = \frac{N}{V} = \text{no. of fermions per unit volume}$$

→ Fermi Temperature (T_F): is defined by the following relation

$$T_F = \frac{\epsilon_F}{k_B} = \frac{h^2}{2mk_B} \left(\frac{3n}{4\pi g_s} \right)^{2/3}$$

if we say.

$$\rho = \text{density of fermi gas} = \frac{mN}{V} = mn.$$

$$\therefore T_F = \frac{h^2}{2m^{5/3} k_B} \left(\frac{3\rho}{4\pi g_s} \right)^{2/3}$$

Fermi Temperature are of the order of $10^4 - 10^5$ K.

• cases:

i) When $T \ll T_F$ or $kT \ll \epsilon_F$, the Fermi Dirac distribution is strongly degenerate

ii) When $T > T_F$ or $kT > \epsilon_F$, the distribution is slightly or weakly degenerate.

iii) When $T \gg T_F$ or $kT \gg \epsilon_F$, the distribution is non-degenerate.

Total Energy of fermions.:

This is the avg energy of fermions at temp $T=0K$

$$E = \int_0^{\infty} E \, dn(E)$$

$$= \int_0^{\infty} E \cdot g_s \frac{4\pi mV}{h^3} \sqrt{2m} E^{1/2} f(E) dE$$

$$= \int_0^{E_F} g_s \frac{4\pi mV}{h^3} \sqrt{2m} E^{3/2} dE \quad \left\{ \begin{array}{l} \text{as} \\ f(E) = 0 \text{ for } E > E_F \end{array} \right.$$

$$= g_s \frac{4\pi mV}{h^3} \sqrt{2m} \frac{2}{5} E_F^{5/2}$$

$$E = \frac{4\pi g_s V (2m)^{3/2}}{5 h^3} E_F^{3/2} (E_F)$$

Since, ~~E_F~~ $N = \frac{4\pi g_s V (2m)^{3/2}}{3 h^3} E_F^{3/2}$

so we can write.

$$E = \frac{3}{5} N E_F$$

The mean energy per fermion for a complete degenerate electron gas is given by

$$\bar{E} = \frac{E}{N} = \frac{3}{5} E_F$$

$$\bar{E} = \frac{3}{5} E_F$$

→ Electron Gas in Metals:

Metals are very good conductors. The high conductivity of metals is due to the presence of free electrons. The free electrons inside a metallic conductor move freely within the metal. They continuously collide with the fixed atoms and thus behave like an electron gas. Since ~~they~~ electrons have half-integral spin angular momentum in units of $\frac{h}{2\pi}$ and they are assumed to obey Pauli's exclusion principle, they should obey the Fermi-Dirac Statistics.

Thus distribution of energies among the electrons in the electron gas in a metal is derived by Fermi-Dirac statistics.

✓ for electron, $s = \frac{1}{2}$ so $g = 2s + 1 = 2$

Therefore

i) Fermi Energy at 0K is

$$E_F = \frac{h^2}{2m} \left(\frac{3N}{4\pi V g s} \right)^{2/3} = \frac{h^2}{2m} \left(\frac{3N}{8\pi V} \right)^{2/3}$$

$$\boxed{E_F = \frac{h^2}{8m} \left(\frac{3N}{\pi V} \right)^{2/3}} = \frac{h^2}{2m} (3\pi^2 n)^{2/3} \quad n = \frac{N}{V}$$

ii) Fermi Temperature (T_F)

$$T_F = \frac{E_F}{k_B} = \frac{h^2}{8mk_B} \left(\frac{3N}{\pi V} \right)^{2/3}$$

$$= \frac{h^2}{8m^{5/3} k_B} \left(\frac{3\rho}{\pi} \right)^{2/3} = 4.52 \times 10^5 \rho^{2/3} \text{ K}$$

$$\boxed{T_F = 4.52 \times 10^5 \rho^{2/3} \text{ K}}$$

where, ρ = density of valence electron.

iii) Total energy (E)

$$E = \frac{3}{5} N \epsilon_f = \frac{3}{5} \frac{h^2}{2m}$$

$$E = \frac{8\pi V}{5h^3} (2m)^{3/2} \epsilon_f^{5/2}$$

iv) Fermi Pressure (P_f)

This is the pressure exerted by fermion gas at $T=0K$

we know that

$$P_f = \frac{2}{3} \left(\frac{E}{V} \right)$$

where, ground state total energy is given by

$$E = \frac{3}{5} N \epsilon_f$$

$$\therefore P_f = \left(\frac{2}{3} \right) \frac{1}{V} \left(\frac{3}{5} N \epsilon_f \right)$$

$$P_f = \frac{2}{5} \left(\frac{N}{V} \epsilon_f \right)$$

This shows that if electron in metal were neutral they would exert a pressure of almost 10^6 atmosphere. Does it make us experience this enormous pressure? No, actually this pressure is counter balanced by the ~~coulomb~~ Coulomb attraction of electron by ions.

v) Graph of n_e vs. ϵ :

Let us consider an electron gas having N electrons in a conductor whose volume is V .

Then, no. of states beth energy ϵ and $\epsilon + d\epsilon$ is

$$g(\epsilon) d\epsilon = \frac{8\pi V m}{h^3} \sqrt{2m\epsilon} d\epsilon$$

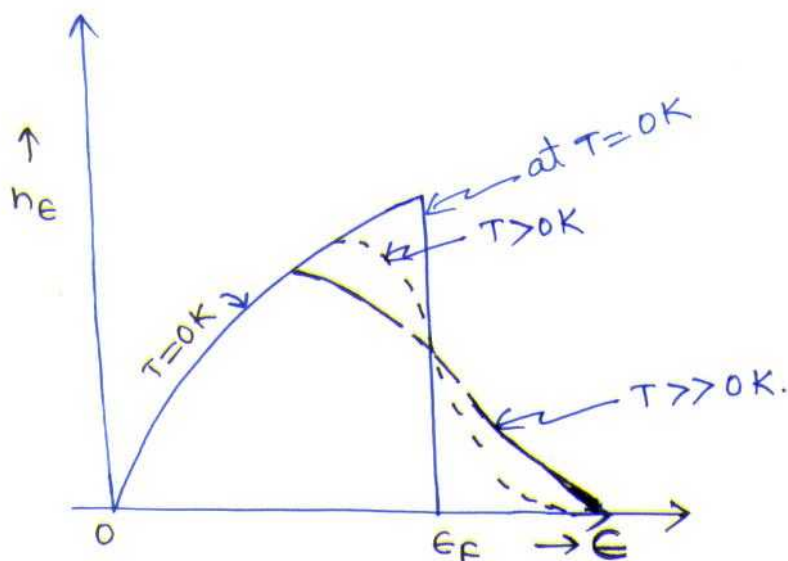
$$dn(\epsilon) = f(\epsilon) g(\epsilon) d\epsilon.$$

$$\text{Since } f(\epsilon) = \frac{1}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

$$\text{So, } dn(\epsilon) = \frac{8\pi V m \sqrt{2m\epsilon}}{h^3} \frac{d\epsilon}{e^{(\epsilon - \epsilon_F)/kT} + 1}$$

$$\text{Since } dn(\epsilon) = n_\epsilon d\epsilon$$

$$\therefore n_\epsilon = \frac{8\pi V m \sqrt{2m\epsilon}}{h^3} \frac{1}{e^{\epsilon - \epsilon_F/kT} + 1}$$



[graph of n_ϵ and ϵ]

at $T=0$

$$n_\epsilon = 0 \text{ for } \epsilon > \epsilon_F.$$

So at $T=0K$, there is complete cut off beyond $\epsilon = \epsilon_F$. ~~At~~ ^{little} At a higher temperature ($T > 0K$),

n_ϵ decreases rapidly for $\epsilon > \epsilon_F$.

But for very high temperatures, the cut off is not sharp.

$S \rightarrow Nk_B \ln 2$. It can be seen that entropy is also a function of $\frac{B}{T}$. A reversible adiabatic demagnetisation is an isentropic process, so that $\frac{B}{T}$ remains constant. Thus, T decreases as B decreases.

Problem 13.9 In the high temperature/low field limit, the entropy can be expressed as $S = k_B \ln(2^N)$. Explain the significance of the factor 2^N in this expression.

13.6.1 Negative Temperatures

The concept of negative absolute temperature seems physically unacceptable. But we will now show that it is possible in principle to realise a state where increase in energy is accompanied by decrease in entropy. Such a state is said to correspond to a negative temperature.

To appreciate the point, let us consider a freely moving molecule of a perfect gas or a harmonic oscillator characterised by an infinite number of energy levels. As temperature of such a system is increased, more molecules will gradually go to higher energy levels. So the system becomes more energetic and greater disorder will prevail. We can therefore say that the entropy associated with the ordinary translational and vibrational degrees of freedom of a body increases without limit as energy increases. In such cases, the ratio $\frac{dU}{dS}$ will always be positive definite. We may therefore conclude that a system with infinite energy levels always stays in positive temperature states.

Let us now again refer to the two-level system of magnetic dipoles with parallel and anti-parallel spins. In an external magnetic field $\mathbf{B}$, these levels have energies $\varepsilon_1 = -\mu_B B$ and $\varepsilon_2 = +\mu_B B$, respectively. Suppose that the number of atoms whose moments are parallel and anti-parallel to $\mathbf{B}$ are N_1 and N_2 , respectively. In the equilibrium state at a temperature T , the average occupation numbers of these levels are given by

$$N_1 = \frac{N}{Z} \exp\left(-\frac{\varepsilon_1}{k_B T}\right)$$

and

$$N_2 = \frac{N}{Z} \exp\left(-\frac{\varepsilon_2}{k_B T}\right)$$

where $N = N_1 + N_2$ denotes the total number of atoms. On combining these expressions, we can write

$$\frac{N_1}{N_2} = \exp\left(\frac{\varepsilon_2 - \varepsilon_1}{k_B T}\right) \quad (13.103)$$

On taking natural logarithm and rearranging terms, we can define temperature in terms of the energies of the levels and their atomic occupancy as

$$T = \frac{1}{k_B} \left(\frac{\varepsilon_2 - \varepsilon_1}{\ln N_1 - \ln N_2} \right) \quad (13.104a)$$

13.42 Thermal Physics

Since $\epsilon_2 > \epsilon_1$, we note that T will be positive if $N_1 > N_2$. This situation is depicted schematically in Fig. 13.19a. The lengths of the vertical lines signify level of occupancy of a state.

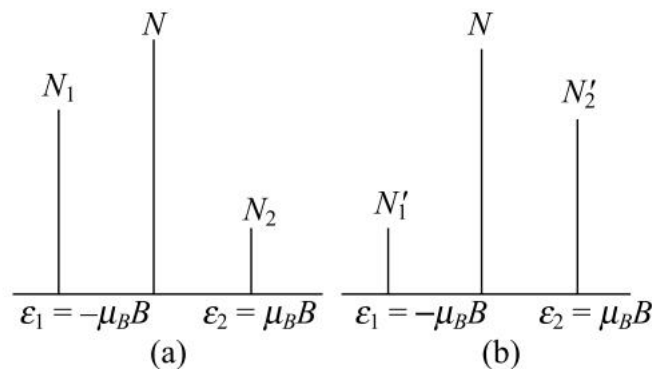


Fig. 13.19 (a) In thermal equilibrium, $N_1 > N_2$ and (b) Reversal of the direction of $\mathbf{B}$ leads to population inversion, $N_2 > N_1$.

Next we suddenly reverse the direction of magnetic field. The dipoles oriented parallel to the field and having energy ϵ_1 will be oriented anti-parallel to the field and have a higher energy. But the dipoles originally anti-parallel to the field and having energy ϵ_2 will now be oriented parallel to the field and have a lower energy. (Eventually, of course, the dipoles in the higher energy state will flip over to the new lower-energy state.) The situation immediately after the reversal of the magnetic field is shown in Fig. 13.19b. Let the average occupation numbers of the new lower and higher energy states be N'_1 and N'_2 , respectively. Then $N'_1 = N_2$ and $N'_2 > N'_1$. This process is referred to as *population inversion*. The temperature corresponding to the situation illustrated in Fig. 13.19b can be mathematically represented by the relation

$$T' = \frac{1}{k_B} \left(\frac{\epsilon_2 - \epsilon_1}{\ln N'_1 - \ln N'_2} \right) \quad (13.104b)$$

Since $N'_2 > N'_1$, this state will correspond to negative T .

Now refer to Fig. 13.20a. It shows the variation of entropy with energy.

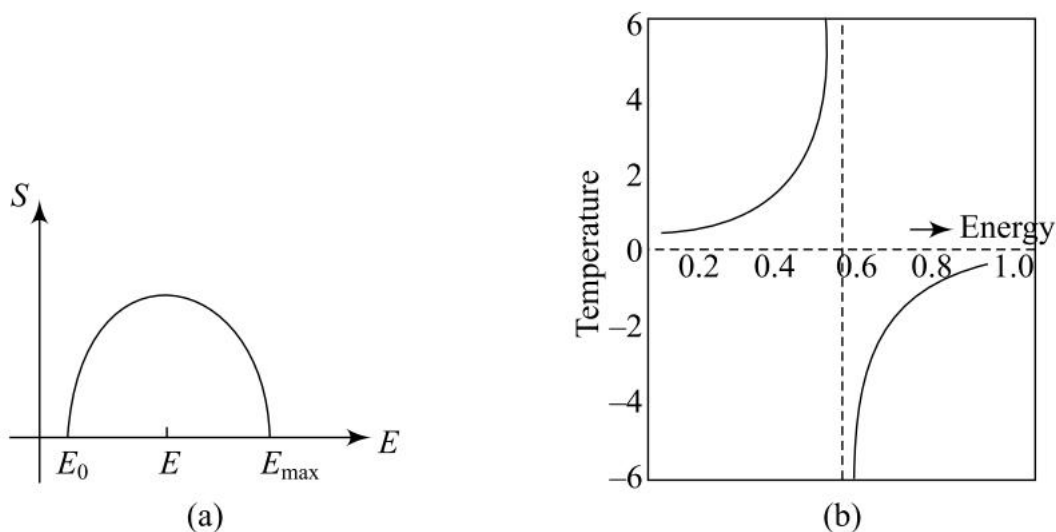


Fig. 13.20 (a) Plot of entropy as a function of internal energy for a two-level system: $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon$ and (b) Plot of temperature as a function of internal energy.

1. From Eq. (13.93a), we recall that for $T = 0$, the probability of occupancy P_1 will be one; i.e., all the atoms will be in the energy state $\epsilon_1 = 0$: $N_1 = N$ and $N_2 = 0$. The internal energy of the system will be zero. This is a state of minimum disorder and corresponds to $S = 0$.
2. As T increases, the occupancy in the higher level begins to rise. As $T \rightarrow \infty$, the probabilities of occupancy P_1 and P_2 will be one-half, i.e., both levels are equally occupied: $N_1 = N_2 = \frac{N}{2}$, and internal energy will be $\frac{N\epsilon}{2}$. This corresponds to state of maximum disorder and hence maximum entropy. In the region $0 < U \leq \frac{N\epsilon}{2}$, the derivatives of S with respect to U will be positive and we can define a positive temperature.
3. If more atoms tend to occupy the higher energy state such that $N_2 > N_1$, we say that population inversion has been attained. In the particular case, if $N_2 = N$, the internal energy will be $N\epsilon$. This corresponds to the state of minimum disorder and zero entropy. Eq. (13.104a) then implies that T will be negative. That is, in the region $\frac{N\epsilon}{2} \leq U \leq N\epsilon$, the slope of entropy-energy curve is negative. This corresponds to the state of *negative temperature*.

We can understand the concept of negative temperatures in yet another way. This is illustrated in Fig. 13.20b, where we have plotted T versus U . Note that as energy increases from zero, T increases from zero and reaches a maximum of ∞ when $U = \frac{N\epsilon}{2}$. As U is infinitesimally greater than $\frac{N\epsilon}{2}$, T abruptly changes to $-\infty$ and when $U = N\epsilon$, we say that it gradually goes to -0 K. We thus arrive at a paradoxical result: *a system at a negative temperature is hotter than infinite absolute temperature!* This means that -0 K is the hottest temperature possible. However, note that the negative temperature is ‘not below absolute zero’ but above infinite absolute temperature.

Experimental Demonstration of Negative Temperatures In paramagnetic substances, the interactions between atomic magnetic moments (the unpaired electrons) and the crystal lattice are very strong and it is not possible for a substance to exist in the state of population inversion for appreciable time. However, in 1951, Pound, Purcell and Ramsey established the existence of negative absolute temperature by using nuclear magnetic resonance techniques. They observed that the nuclear magnetic moments of the Li atoms in LiF interacted extremely slowly with the lattice and quite a long time, up to several minutes, elapsed before they attained equilibrium with the lattice. As a result, the system existed in the state of population inversion, i.e., the state of negative temperature, for quite some time.

In 1917, Einstein had predicted that population inversion can occur due to transitions between different energy states in an atom with finite number of energy levels when he gave the theory of stimulated emission. It was realised in 1954 by C. H. Townes and his co-workers for *microwaves*. But Maiman operated the first successful laser in 1960 using a ruby crystal.

Planck units

In particle physics and physical cosmology, **Planck units** are a set of units of measurement defined exclusively in terms of four universal physical constants, in such a manner that these physical constants take on the numerical value of 1 when expressed in terms of these units. Originally proposed in 1899 by German physicist Max Planck, these units are a system of natural units because their definition is based on properties of nature, more specifically the properties of free space, rather than a choice of prototype object. They are relevant in research on unified theories such as quantum gravity.

The term **Planck scale** refers to quantities of space, time, energy and other units that are similar in magnitude to corresponding Planck units. This region may be characterized by particle energies of around 10^{19} GeV, time intervals of around 10^{-43} s and lengths of around 10^{-35} m (approximately the energy-equivalent of the Planck mass, the Planck time and the Planck length, respectively). At the Planck scale, the predictions of the Standard Model, quantum field theory and general relativity are not expected to apply, and quantum effects of gravity are expected to dominate. The best-known example is represented by the conditions in the first 10^{-43} seconds of our universe after the Big Bang, approximately 13.8 billion years ago.

The four universal constants that, by definition, have a numeric value 1 when expressed in these units are:

- the speed of light in vacuum, c ,
- the gravitational constant, G ,
- the reduced Planck constant, $\hbar$,
- the Boltzmann constant, k_B .

Planck units do not incorporate an electromagnetic dimension. Some authors choose to extend the system to electromagnetism by, for example, adding either the electric constant ϵ_0 or $4\pi\epsilon_0$ to this list. Similarly, authors choose to use variants of the system that give other numeric values to one or more of the four constants above.

Contents

[Introduction](#)

[History and definition](#)

[Derived units](#)

[Significance](#)

Planck scale

Relationship to gravity

In cosmology

Analysis of the units

Planck length

Planck time

Planck energy

Planck unit of force

Planck temperature

Nondimensionalized equations

Alternative choices of normalization

Gravitational constant

See also

Explanatory notes

References

External links

Introduction

Any system of measurement may be assigned a mutually independent set of base quantities and associated base units, from which all other quantities and units may be derived. In the International System of Units, for example, the SI base quantities include length with the associated unit of the metre. In the system of Planck units, a similar set of base quantities and associated units may be selected, in terms of which other quantities and coherent units may be expressed.^{[1][2]:1215} The Planck unit of length has become known as the Planck length, and the Planck unit of time is known as the Planck time, but this nomenclature has not been established as extending to all quantities.

All Planck units are derived from the dimensional universal physical constants that define the system, and in a convention in which these units are omitted (i.e. treated as having the dimensionless value 1), these constants are then eliminated from equations of physics in which they appear. For example, Newton's law of universal gravitation,

$$F = G \frac{m_1 m_2}{r^2} = \left(\frac{F_{\text{P}} l_{\text{P}}^2}{m_{\text{P}}^2} \right) \frac{m_1 m_2}{r^2},$$

can be expressed as:

$$\frac{F}{F_P} = \frac{\left(\frac{m_1}{m_P}\right) \left(\frac{m_2}{m_P}\right)}{\left(\frac{r}{l_P}\right)^2}.$$

Both equations are dimensionally consistent and equally valid in *any* system of quantities, but the second equation, with G absent, is relating only dimensionless quantities since any ratio of two like-dimensioned quantities is a dimensionless quantity. If, by a shorthand convention, it is understood that each physical quantity is the corresponding ratio with a coherent Planck unit (or "expressed in Planck units"), the ratios above may be expressed simply with the symbols of physical quantity, without being scaled explicitly by their corresponding unit:

$$F' = \frac{m'_1 m'_2}{r'^2}.$$

This last equation (without G) is valid with F' , m'_1 , m'_2 , and r' being the dimensionless ratio quantities corresponding to the standard quantities, written e.g. $F' \triangleq F$ or $F' = F/F_P$, but not as a direct equality of quantities. This may seem to be "setting the constants c , G , etc., to 1" if the correspondence of the quantities is thought of as equality. For this reason, Planck or other natural units should be employed with care. Referring to " $G = c = 1$ ", Paul S. Wesson wrote that, "Mathematically it is an acceptable trick which saves labour. Physically it represents a loss of information and can lead to confusion."^[3]

History and definition

The concept of natural units was introduced in 1874, when George Johnstone Stoney, noting that electric charge is quantized, derived units of length, time, and mass, now named Stoney units in his honor. Stoney chose his units so that G , c , and the electron charge e would be numerically equal to 1.^[4] In 1899, one year before the advent of quantum theory, Max Planck introduced what became later known as the Planck constant.^{[5][6]} At the end of the paper, he proposed the base units that were later named in his honor. The Planck units are based on the quantum of action, now usually known as the Planck constant, which appeared in the Wien approximation for black-body radiation. Planck underlined the universality of the new unit system, writing:

... die Möglichkeit gegeben ist, Einheiten für Länge, Masse, Zeit und Temperatur aufzustellen, welche, unabhängig von speciellen Körpern oder Substanzen, ihre Bedeutung für alle Zeiten und für alle, auch außerirdische und außermenschliche Culturen notwendig behalten und welche daher als »natürliche Maßeinheiten« bezeichnet werden können.

... it is possible to set up units for length, mass, time and temperature, which

are independent of special bodies or substances, necessarily retaining their meaning for all times and for all civilizations, including extraterrestrial and non-human ones, which can be called "natural units of measure".

Planck considered only the units based on the universal constants G , h , c , and k_B to arrive at natural units for length, time, mass, and temperature.^[6] His definitions differ from the modern ones by a factor of $\sqrt{2\pi}$, because the modern definitions use $\hbar$ rather than h .^{[5][6]}

Table 1: Modern values for Planck's original choice of quantities

Name	Dimension	Expression	Value (SI units)
Planck length	<u>length</u> (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	$1.616\,255(18) \times 10^{-35} \text{ m}^{[7]}$
Planck mass	<u>mass</u> (M)	$m_P = \sqrt{\frac{\hbar c}{G}}$	$2.176\,434(24) \times 10^{-8} \text{ kg}^{[8]}$
Planck time	<u>time</u> (T)	$t_P = \sqrt{\frac{\hbar G}{c^5}}$	$5.391\,247(60) \times 10^{-44} \text{ s}^{[9]}$
Planck temperature	<u>temperature</u> (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}}$	$1.416\,784(16) \times 10^{32} \text{ K}^{[10]}$

Unlike the case with the International System of Units, there is no official entity that establishes a definition of a Planck unit system. Some authors define the base Planck units to be those of mass, length and time, regarding an additional unit for temperature to be redundant.^[note 1] Other tabulations add, in addition to a unit for temperature, a unit for electric charge, so that the vacuum permittivity ϵ_0 is also normalized to 1.^{[12][13]} Some of these tabulations also replace mass with energy when doing so.^[14] Depending on the author's choice, this charge unit is given by

$$q_P = \sqrt{4\pi\epsilon_0 \hbar c} \approx 1.875546 \times 10^{-18} \text{ C} \approx 11.7 \text{ } e$$

or

$$q_P = \sqrt{\epsilon_0 \hbar c} \approx 5.290818 \times 10^{-19} \text{ C} \approx 3.3 \text{ } e.$$

The Planck charge, as well as other electromagnetic units that can be defined like resistance and magnetic flux, are more difficult to interpret than Planck's original units and are used less frequently.^[15]

In SI units, the values of c , h , e and k_B are exact and the values of ϵ_0 and G in SI units respectively have relative uncertainties of 1.5×10^{-10} ^[16] and 2.2×10^{-5} .^[17] Hence, the uncertainties in the SI values of the Planck units derive almost entirely from uncertainty

in the SI value of G .

Derived units

In any system of measurement, units for many physical quantities can be derived from base units. Table 2 offers a sample of derived Planck units, some of which are seldom used. As with the base units, their use is mostly confined to theoretical physics because most of them are too large or too small for empirical or practical use and there are large uncertainties in their values.

Table 2: Coherent derived units of Planck units

Derived unit of	Expression	Approximate <u>SI</u> equivalent
<u>area</u> (L^2)	$l_P^2 = \frac{\hbar G}{c^3}$	$2.6121 \times 10^{-70} \text{ m}^2$
<u>volume</u> (L^3)	$l_P^3 = \left(\frac{\hbar G}{c^3}\right)^{\frac{3}{2}} = \sqrt{\frac{(\hbar G)^3}{c^9}}$	$4.2217 \times 10^{-105} \text{ m}^3$
<u>momentum</u> (LMT^{-1})	$m_P c = \frac{\hbar}{l_P} = \sqrt{\frac{\hbar c^3}{G}}$	$6.5249 \text{ kg} \cdot \text{m/s}$
<u>energy</u> (L^2MT^{-2})	$E_P = m_P c^2 = \frac{\hbar}{t_P} = \sqrt{\frac{\hbar c^5}{G}}$	$1.9561 \times 10^9 \text{ J}$
<u>force</u> (LMT^{-2})	$F_P = \frac{E_P}{l_P} = \frac{\hbar}{l_P t_P} = \frac{c^4}{G}$	$1.2103 \times 10^{44} \text{ N}$
<u>density</u> ($L^{-3}M$)	$\rho_P = \frac{m_P}{l_P^3} = \frac{\hbar t_P}{l_P^5} = \frac{c^5}{\hbar G^2}$	$5.1550 \times 10^{96} \text{ kg/m}^3$
<u>acceleration</u> (LT^{-2})	$a_P = \frac{c}{t_P} = \sqrt{\frac{c^7}{\hbar G}}$	$5.5608 \times 10^{51} \text{ m/s}^2$

Some Planck units, such as of time and length, are many orders of magnitude too large or too small to be of practical use, so that Planck units as a system are typically only relevant to theoretical physics. In some cases, a Planck unit may suggest a limit to a range of a physical quantity where present-day theories of physics apply.^[18] For example, our understanding of the Big Bang does not extend to the Planck epoch, i.e., when the universe was less than one Planck time old. Describing the universe during the Planck epoch requires a theory of quantum gravity that would incorporate quantum effects into general relativity. Such a theory does not yet exist.

Several quantities are not "extreme" in magnitude, such as the Planck mass, which is about 22 micrograms: very large in comparison with subatomic particles, and within the mass range of living organisms.^{[19]:872} Similarly, the related units of energy and of

momentum are in the range of some everyday phenomena.

Significance

Planck units have little anthropocentric arbitrariness, but do still involve some arbitrary choices in terms of the defining constants. Unlike the metre and second, which exist as base units in the SI system for historical reasons, the Planck length and Planck time are conceptually linked at a fundamental physical level. Consequently, natural units help physicists to reframe questions. Frank Wilczek puts it succinctly:

We see that the question [posed] is not, "Why is gravity so feeble?" but rather, "Why is the proton's mass so small?" For in natural (Planck) units, the strength of gravity simply is what it is, a primary quantity, while the proton's mass is the tiny number $[1/(13 \text{ quintillion})]$.^[20]

While it is true that the electrostatic repulsive force between two protons (alone in free space) greatly exceeds the gravitational attractive force between the same two protons, this is not about the relative strengths of the two fundamental forces. From the point of view of Planck units, this is comparing apples with oranges, because mass and electric charge are incommensurable quantities. Rather, the disparity of magnitude of force is a manifestation of the fact that the charge on the protons is approximately the unit charge but the mass of the protons is far less than the unit mass.

Planck scale

In particle physics and physical cosmology, the Planck scale is an energy scale around 1.22×10^{19} GeV (the Planck energy, corresponding to the energy equivalent of the Planck mass, is $2.176 45 \times 10^{-8}$ kg) at which quantum effects of gravity become significant. At this scale, present descriptions and theories of sub-atomic particle interactions in terms of quantum field theory break down and become inadequate, due to the impact of the apparent non-renormalizability of gravity within current theories.

Relationship to gravity

At the Planck length scale, the strength of gravity is expected to become comparable with the other forces, and it has been theorized that all the fundamental forces are unified at that scale, but the exact mechanism of this unification remains unknown.^[21] The Planck scale is therefore the point at which the effects of quantum gravity can no longer be ignored in other fundamental interactions, where current calculations and approaches begin to break down, and a means to take account of its impact is necessary.^[22] On these grounds, it has been speculated that it may be an approximate lower limit at which a black hole could be formed by collapse.^[23]

While physicists have a fairly good understanding of the other fundamental interactions of forces on the quantum level, gravity is problematic, and cannot be integrated with quantum mechanics at very high energies using the usual framework of quantum field theory. At lesser energy levels it is usually ignored, while for energies approaching or exceeding the Planck scale, a new theory of quantum gravity is necessary. Approaches to this problem include string theory and M-theory, loop quantum gravity, noncommutative geometry, and causal set theory.

In cosmology

In Big Bang cosmology, the Planck epoch or Planck era is the earliest stage of the Big Bang, before the time passed was equal to the Planck time, t_P , or approximately 10^{-43} seconds.^[24] There is no currently available physical theory to describe such short times, and it is not clear in what sense the concept of time is meaningful for values smaller than the Planck time. It is generally assumed that quantum effects of gravity dominate physical interactions at this time scale. At this scale, the unified force of the Standard Model is assumed to be unified with gravitation. Immeasurably hot and dense, the state of the Planck epoch was succeeded by the grand unification epoch, where gravitation is separated from the unified force of the Standard Model, in turn followed by the inflationary epoch, which ended after about 10^{-32} seconds (or about $10^{11} t_P$).^[25]

Table 3 lists properties of the observable universe today expressed in Planck units.^{[26][27]}

Table 3: Today's universe in Planck units

Property of present-day observable universe	Approximate number of Planck units	Equivalents
Age	$8.08 \times 10^{60} t_P$	4.35×10^{17} s or 1.38×10^{10} years
Diameter	$5.4 \times 10^{61} l_P$	8.7×10^{26} m or 9.2×10^{10} light-years
Mass	approx. $10^{60} m_P$	3×10^{52} kg or 1.5×10^{22} solar masses (only counting stars) 10^{80} protons (sometimes known as the Eddington number)
Density	$1.8 \times 10^{-123} m_P \cdot l_P^{-3}$	9.9×10^{-27} kg·m ⁻³
Temperature	$1.9 \times 10^{-32} T_P$	2.725 K temperature of the cosmic microwave background radiation
Cosmological constant	$\approx 10^{-122} l_P^{-2}$	$\approx 10^{-52}$ m ⁻²
Hubble constant	$\approx 10^{-61} t_P^{-1}$	$\approx 10^{-18}$ s ⁻¹ $\approx 10^2$ (km/s)/Mpc

After the measurement of the cosmological constant (Λ) in 1998, estimated at 10^{-122} in Planck units, it was noted that this is suggestively close to the reciprocal of the age of the universe (T) squared. Barrow and Shaw proposed a modified theory in which Λ is a field evolving in such a way that its value remains $\Lambda \sim T^{-2}$ throughout the history of the universe.^[28]

Analysis of the units

Planck length

The Planck length, denoted ℓ_P , is a unit of length defined as:

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}}$$

It is equal to $1.616\,255(18) \times 10^{-35}$ m,^[7] where the two digits enclosed by parentheses are the estimated standard error associated with the reported numerical value, or about 10^{-20} times the diameter of a proton.^[29] It can be motivated in various ways, such as considering a particle whose reduced Compton wavelength is comparable to its Schwarzschild radius,^{[29][30][31]} though whether those concepts are in fact simultaneously applicable is open to debate.^[32] (The same heuristic argument simultaneously motivates the Planck mass.^[30])

The Planck length is a distance scale of interest in speculations about quantum gravity. The Bekenstein–Hawking entropy of a black hole is one-fourth the area of its event horizon in units of Planck length squared.^{[11]:370} Since the 1950s, it has been conjectured that quantum fluctuations of the spacetime metric might make the familiar notion of distance inapplicable below the Planck length.^{[33][34][35]} This is sometimes expressed by saying that "spacetime becomes a foam at the Planck scale".^[36] It is possible that the Planck length is the shortest physically measurable distance, since any attempt to investigate the possible existence of shorter distances, by performing higher-energy collisions, would result in black hole production. Higher-energy collisions, rather than splitting matter into finer pieces, would simply produce bigger black holes.^[37]

The strings of string theory are modeled to be on the order of the Planck length.^{[38][39]} In theories with large extra dimensions, the Planck length calculated from the observed value of G can be smaller than the true, fundamental Planck length.^{[11]:61[40]}

Planck time

The Planck time t_P is the time required for light to travel a distance of 1 Planck length in vacuum, which is a time interval of approximately 5.39×10^{-44} s. No current physical theory can describe timescales shorter than the Planck time, such as the earliest events after the Big Bang.^[24] Some conjecture that the structure of time need not remain smooth on intervals comparable to the Planck time.^[41]

Planck energy

Most Planck units are extremely small, as in the case of Planck length or Planck time, or extremely large, as in the case of Planck temperature or Planck density. For comparison, the Planck energy E_P is approximately equal to the energy released in the combustion of the fuel in an automobile fuel tank (57.2 L at 34.2 MJ/L of chemical energy). The ultra-high-energy cosmic ray observed in 1991 had a measured energy of about 50 J, equivalent to about $2.5 \times 10^{-8} E_P$.^{[42][43]}

Proposals for theories of doubly special relativity posit that, in addition to the speed of light, an energy scale is also invariant for all inertial observers. Typically, this energy scale is chosen to be the Planck energy.^{[44][45]}

Planck unit of force

The Planck unit of force may be thought of as the derived unit of force in the Planck system if the Planck units of time, length, and mass are considered to be base units.

$$F_P = \frac{m_P c}{t_P} = \frac{c^4}{G} \approx 1.2103 \times 10^{44} \text{ N}$$

It is the gravitational attractive force of two bodies of 1 Planck mass each that are held 1 Planck length apart. One convention for the Planck charge is to choose it so that the electrostatic repulsion of two objects with Planck charge and mass that are held 1 Planck length apart balances the Newtonian attraction between them.^[46]

Some authors have argued that the Planck force is on the order of the maximum force that can occur between two bodies.^{[47][48]} However, the validity of these conjectures has been disputed.^{[49][50]}

Planck temperature

The Planck temperature T_P is $1.416\,784(16) \times 10^{32}$ K.^[10] At this temperature, the wavelength of light emitted by thermal radiation reaches the Planck length. There are no known physical models able to describe temperatures greater than T_P ; a quantum theory of gravity would be required to model the extreme energies attained.^[51] Hypothetically, a system in thermal equilibrium at the Planck temperature might contain Planck-scale

black holes, constantly being formed from thermal radiation and decaying via Hawking evaporation. Adding energy to such a system might *decrease* its temperature by creating larger black holes, whose Hawking temperature is lower.^[52]

Nondimensionalized equations

Physical quantities that have different dimensions (such as time and length) cannot be equated even if they are numerically equal (e.g., 1 second is not the same as 1 metre). In theoretical physics, however, this scruple may be set aside, by a process called nondimensionalization. The effective result is that many fundamental equations of physics, which often include some of the constants used to define Planck units, become equations where these constants are replaced by a 1.

Examples include the energy–momentum relation $E^2 = (mc^2)^2 + (pc)^2$, which becomes $E^2 = m^2 + p^2$, and the Dirac equation $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$, which becomes $(i\gamma^\mu\partial_\mu - m)\psi = 0$.

Alternative choices of normalization

As already stated above, Planck units are derived by "normalizing" the numerical values of certain fundamental constants to 1. These normalizations are neither the only ones possible nor necessarily the best. Moreover, the choice of what factors to normalize, among the factors appearing in the fundamental equations of physics, is not evident, and the values of the Planck units are sensitive to this choice.

The factor 4π is ubiquitous in theoretical physics because in three-dimensional space, the surface area of a sphere of radius r is $4\pi r^2$. This, along with the concept of flux, are the basis for the inverse-square law, Gauss's law, and the divergence operator applied to flux density. For example, gravitational and electrostatic fields produced by point objects have spherical symmetry, and so the electric flux through a sphere of radius r around a point charge will be distributed uniformly over that sphere. From this, it follows that a factor of $4\pi r^2$ will appear in the denominator of Coulomb's law in rationalized form.^{[26]:214–15} (Both the numerical factor and the power of the dependence on r would change if space were higher-dimensional; the correct expressions can be deduced from the geometry of higher-dimensional spheres.^{[11]:51}) Likewise for Newton's law of universal gravitation: a factor of 4π naturally appears in Poisson's equation when relating the gravitational potential to the distribution of matter.^{[11]:56}

Hence a substantial body of physical theory developed since Planck's 1899 paper suggests normalizing not G but $4\pi G$ (or $8\pi G$) to 1. Doing so would introduce a factor of $\frac{1}{4\pi}$ (or $\frac{1}{8\pi}$) into the nondimensionalized form of the law of universal gravitation, consistent with the modern rationalized formulation of Coulomb's law in terms of the vacuum permittivity. In fact, alternative normalizations frequently preserve the factor of $\frac{1}{4\pi}$ in the nondimensionalized form of Coulomb's law as well, so that the nondimensionalized

Maxwell's equations for electromagnetism and gravitoelectromagnetism both take the same form as those for electromagnetism in SI, which do not have any factors of 4π . When this is applied to electromagnetic constants, ϵ_0 , this unit system is called "*rationalized*". When applied additionally to gravitation and Planck units, these are called **rationalized Planck units**^[53] and are seen in high-energy physics.^[54]

The rationalized Planck units are defined so that $c = 4\pi G = \hbar = \epsilon_0 = k_B = 1$.

There are several possible alternative normalizations.

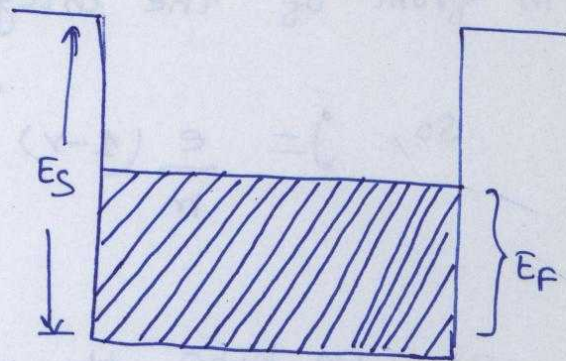
Gravitational constant

In 1899, Newton's law of universal gravitation was still seen as exact, rather than as a convenient approximation holding for "small" velocities and masses (the approximate nature of Newton's law was shown following the development of general relativity in 1915). Hence Planck normalized to 1 the gravitational constant G in Newton's law. In theories emerging after 1899, G nearly always appears in formulae multiplied by 4π or a small integer multiple thereof. Hence, a choice to be made when designing a system of natural units is which, if any, instances of 4π appearing in the equations of physics are to be eliminated via the normalization.

- Normalizing $4\pi G$ to 1 (and therefore setting $G = \frac{1}{4\pi}$):
 - Gauss's law for gravity becomes $\Phi_g = -M$ (rather than $\Phi_g = -4\pi M$ in Planck units).
 - Eliminates $4\pi G$ from the Poisson equation.
 - Eliminates $4\pi G$ in the gravitoelectromagnetic (GEM) equations, which hold in weak gravitational fields or locally flat spacetime. These equations have the same form as Maxwell's equations (and the Lorentz force equation) of electromagnetism, with mass density replacing charge density, and with $\frac{1}{4\pi G}$ replacing ϵ_0 .
 - Normalizes the characteristic impedance Z_g of gravitational radiation in free space to 1 (normally expressed as $\frac{4\pi G}{c}$).^[note 2]
 - Eliminates $4\pi G$ from the Bekenstein–Hawking formula (for the entropy of a black hole in terms of its mass m_{BH} and the area of its event horizon A_{BH}) which is simplified to $S_{BH} = \pi A_{BH} = (m_{BH})^2$.
- Setting $8\pi G = 1$ (and therefore setting $G = \frac{1}{8\pi}$). This would eliminate $8\pi G$ from the Einstein field equations, Einstein–Hilbert action, and the Friedmann equations, for gravitation. Planck units modified so that $8\pi G = 1$ are known as *reduced Planck units*, because the Planck mass is divided by $\sqrt{8\pi}$. Also, the Bekenstein–Hawking formula for the entropy of a black hole simplifies to $S_{BH} = (m_{BH})^2/2 = 2\pi A_{BH}$.

Thermionic Emission from metals:-

Let us define the energy of a free electron at rest inside the metal as zero. In order to escape from the metal an electron must have an energy perpendicular to the surface



of at least E_s . Thus if x is the coordinate perpendicular to the surface, an electron must have a momentum $p_x \geq p_{x0}$ in order to escape, where

$$\frac{p_{x0}^2}{2m} = E_s$$

--- 1)

However, even if an electron at the surface has a momentum $p_x \geq p_{x0}$, it does not necessarily escape, but may be reflected by the potential barrier.

Thus the probability of escape for an electron satisfying the condition $p_x \geq p_{x0}$ is equal to $[1 - r(p_x)]$

where, $r(p_x)$ is reflection coefficient as function of p_x , it is also a function of shape of potential barrier.

Suppose no. of electrons per unit volume with a momentum between p_x and $p_x + dp_x$ inside the metal is equal to $n(p_x) dp_x$.

No. of such electrons arriving at the surface per second per unit area is ~~equal~~ ~~is~~ $= v_x n(p_x) dp_x$

So emission current density is equal to

$$j = \frac{e}{m} \int_{p_{x0}}^{\infty} p_x n(p_x) [1 - r(p_x)] dp_x$$

The term in the bracket is replaced by a factor $(1-r)$ in front of the integral, $r = \text{avg value of reflection coefficient}$

$$\text{So, } j = \frac{e}{m} (1-r) \int_{p_{x0}}^{\infty} p_x n(p_x) dp_x = \dots = 2)$$

as we know, the no. of allowed states / volume corresponding to an element $dp_x dp_y dp_z$ is

$$n(p_x, p_y, p_z) = \frac{2}{h^3} dp_x dp_y dp_z$$

The no. of electrons occupying states with momenta between $p_x, p_x + dp_x$; $p_y, p_y + dp_y$ and $p_z, p_z + dp_z$ is therefore

$$n(p_x, p_y, p_z) dp_x dp_y dp_z = \frac{2}{h^3} \frac{dp_x dp_y dp_z}{1 + e^{(E - E_F)/kT}}$$

$$\text{where, } E = p_x^2 + p_y^2 + p_z^2 / 2m$$

$$\text{So, } n(p_x) dp_x = \frac{2}{h^3} dp_x \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_y dp_z}{e^{(E - E_F)/kT} + 1}$$

since $E - E_F = E_s - E_F \gg kT$ for all metals at room temperature so we can neglect the term of unity in the distribution function.

$$\therefore n(p_x) dp_x = \frac{2}{h^3} \frac{e^{-p_x^2/2mkT}}{e^{-E_F/kT}} dp_x \int_{-\infty}^{+\infty} e^{-p_y^2/2mkT} dp_y \int_{-\infty}^{+\infty} e^{-p_z^2/2mkT} dp_z$$

$$n(p_x) dp_x = \left(\frac{4\pi m kT}{h^3} \right) e^{E_F/kT} e^{-p_x^2/2mkT} dp_x = \dots = 2)$$

putting value of $n(p_x)dp_x$ from eqnⁿ (2) in eqnⁿ 1.), we get

$$j = \frac{e}{m} \left(\frac{4\pi m k T}{h^3} \right) (1-r) \int_{p_{x_0}}^{\infty} p_x e^{-\left(\frac{p_x^2}{2m} - E_F\right)/kT} dp_x.$$

$$= \frac{e}{m} \left(\frac{4\pi m k T}{h^3} \right) (1-r) \cdot (m k T) e^{-\left(\frac{p_{x_0}^2}{2m} - E_F\right)/kT}$$

$$j = \frac{4\pi e m k^2}{h^3} (1-r) T^2 e^{-(E_s - E_F)/kT}$$

Since, $E_s - E_F = \phi$ and, $A = \frac{4\pi e m k^2}{h^3}$

so, $j = A(1-r) T^2 e^{-\phi/kT}$

This is the Dushman-Richardson Equation.

Energy distribution of the emitted electrons:-

According to Dushman eqnⁿ, the total no. of electrons emitted per unit area per second is

$$N = \left(\frac{4\pi m k^2}{h^3} \right) T^2 e^{-\phi/kT} \quad \text{--- 1.)}$$

we assume $r=0$ here.

Also, the no. of electrons arriving at the surface per unit area per second with $p_x \geq p_{x_0}$ and velocities normal to the surface in the range dv_x is

$$n(v_x) v_x dv_x = \left(\frac{4\pi m^2 k T}{h^3} \right) e^{E_F/kT} e^{-m v_x^2 / 2kT} v_x dv_x \quad \text{--- 2.)}$$

WKB Approximation

- Slow varying potentials on the scale of λ ; used to describe tunnelling.

- Condition: $\left| \frac{1}{k(x)} \frac{dk}{dx} \right| \ll k(x)$

- Zeroth Solⁿ: $\psi_0(x) = \exp\left[\pm i \int^x k(x) dx\right]$

- First order: $\psi_1(x) = \frac{C}{\sqrt{k(x)}} \exp\left[\pm i \int^x k(x) dx\right]$

where $k(x) = \sqrt{\frac{2m(E-V)}{\hbar^2}}$

- Connection Formula:

$$\frac{2}{\sqrt{k(x)}} \sin\left[\int_x^a k(x) dx + \frac{\pi}{4}\right] \leftrightarrow \frac{1}{\sqrt{k'(x)}} \exp\left[-\int_a^x k'(x) dx\right]$$

$$\frac{1}{\sqrt{k(x)}} \cos\left[\int_x^a k(x) dx + \frac{\pi}{4}\right] \leftrightarrow \frac{1}{\sqrt{k'(x)}} \exp\left[+\int_a^x k'(x) dx\right]$$

- WKB quantisation condition for parabolic potential

$$\theta = \int_a^b k(x) dx = \left(n + \frac{1}{2}\right)\pi \quad \text{where } a, b \text{ are turning pts}$$

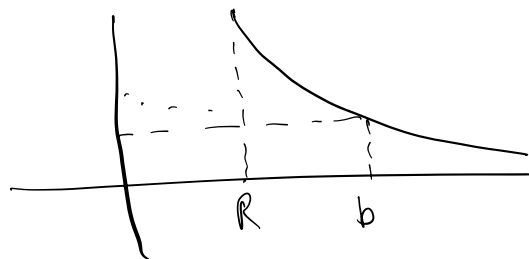
(*) For SHO: $k(x) = \frac{2m}{\hbar^2} \left(E - \frac{1}{2} m \omega^2 x^2\right) \longleftrightarrow \begin{matrix} a = -\sqrt{\frac{2E}{m\omega^2}} \\ b = \sqrt{\frac{2E}{m\omega^2}} \end{matrix}$

$\hookrightarrow$ use $y = \sqrt{\frac{m\omega}{\hbar}} x$ & $y = \sqrt{\lambda} \sin\theta$ in integral.

$$\Rightarrow E = \left(n + \frac{1}{2}\right) \hbar \omega$$

• Tunneling : $T = e^{-2 \int_a^b k'(x) dx}$

• For α particle

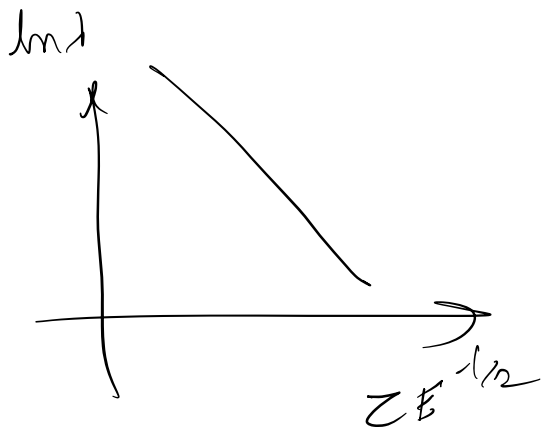


$$V(r) = V_0 \quad r < R$$

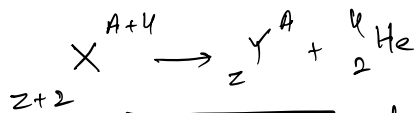
$$= \frac{ZZe^2}{4\pi\epsilon_0 r} \quad r > R$$

$R = \text{radius}$

$$b = \frac{ZZe^2}{4\pi\epsilon_0 E} \quad (\text{2nd turning pt})$$



R in fermi
 E in MeV



$$\lambda = \lambda T \quad (\text{decay constant})$$

Transmission prob
freq. of collision

$$\text{SWE: } \frac{d^2 u}{dr^2} + \frac{2m}{\hbar^2} (E - V(r)) u = 0$$

$$\text{By WKB: } T = \exp\left(-2 \int_R^b \sqrt{\frac{2m}{\hbar^2} (V-E)} dr\right) = e^{-2G}$$

Evaluate integral:

$$G = \int_R^b \sqrt{\frac{2m}{\hbar^2} \left(\frac{ZZe^2}{4\pi\epsilon_0 r} - \frac{ZZe^2}{4\pi\epsilon_0 b} \right)} dr$$

$$= \sqrt{\frac{2m}{\hbar^2} \frac{ZZe^2}{4\pi\epsilon_0}} \int_R^b \sqrt{\frac{1}{r} - \frac{1}{b}} dr$$

$$\text{Let: } r = b \cos^2 \theta$$

$$\Rightarrow G = \sqrt{\frac{2m}{\hbar^2} \frac{ZZe^2}{4\pi\epsilon_0}} (-) \int_{\cos^{-1}(\sqrt{R/b})}^0 \sqrt{b} \cdot 2 \sin^2 \theta d\theta$$

$$(\text{use: } \frac{R}{b} < 1 \text{ \& } \cos^{-1}(\sqrt{\frac{R}{b}}) = \frac{\pi}{2} - \sqrt{\frac{R}{b}})$$

$$\Rightarrow G = \sqrt{\frac{2m}{\hbar^2} \frac{ZZe^2}{4\pi\epsilon_0}} \left(\frac{\pi}{2} - 2\sqrt{\frac{R}{b}} \right)$$

use value of b & $Z = 2$ for α particle

$$\& \text{ get } G = -2 \sqrt{\frac{4me^2}{4\pi\epsilon_0 \hbar^2}} \sqrt{R} \sqrt{Z} + \frac{\pi}{2} \sqrt{\frac{Zme^4}{(4\pi\epsilon_0 \hbar)^2}} Z E^{-1/2}$$

$$\ln T = -2G = A_1 \sqrt{R} \sqrt{Z} + B_1 Z E^{-1/2}$$

$$= 2.97 \sqrt{R} \sqrt{Z} - 3.95 Z E^{-1/2}$$

$$\text{Now } \lambda = \lambda T \quad \left(\lambda = \frac{v_A}{2R} \right)$$

$$\ln \lambda = \ln\left(\frac{v_A}{2R}\right) + 2.97 \sqrt{R} \sqrt{Z} - 3.95 Z E^{-1/2}$$

Geiger Nuttel Law

Fresnel Integral

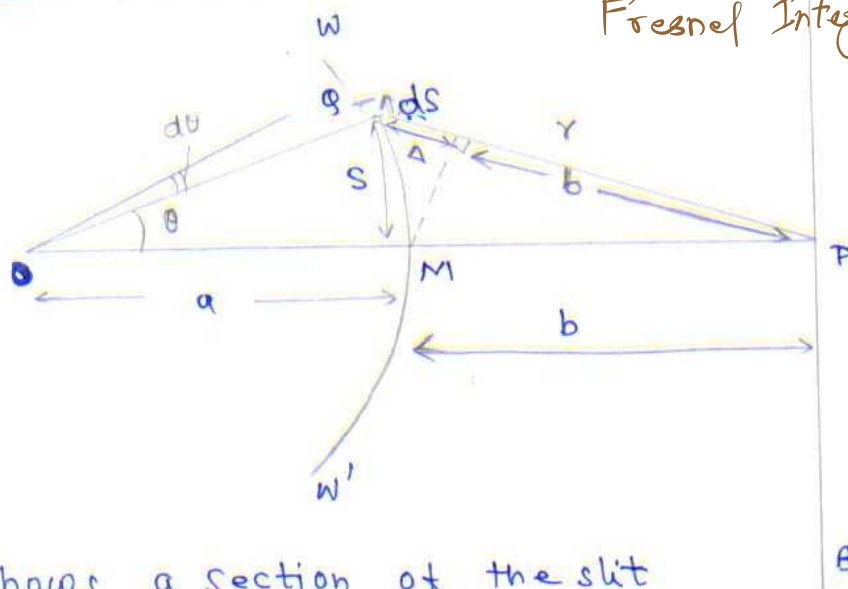


Figure shows a section of the slit

the cylindrical wavefront WW' and the screen AB . The wavelength of the source is λ . M is the pole of the wavefront corresponding to point P .

The half period zones obtained as rectangular strips with their lengths parallel to the slit S .

Consider QR be a narrow strip of the wavefront at a distance s along the arc from M . The width of the strip is

ds . From Huygen's principle, every point on the primary wave-front is a source of secondary wavelets and the resultant intensity at P is combined effect of all secondary wavelets. The contribution of this strip to the disturbance at P at time t is

$$dy = f \sin \{ \omega t - kr \} ds. \quad \text{--- ①}$$

where f is a constant of proportionality, the angle θ is small enough so that the amplitude at Q is proportional to ωt .

Now if $r = b + \Delta$

So eqn ① becomes

$$dy = f \sin \{ \omega t - k(b + \Delta) \} ds$$

$$= f [\sin \{ \omega t - kb \} - \cos \{ \omega t - kb \} k \Delta] ds$$

$$= f \{ \sin(\omega t - kb) \cos k \Delta - \cos(\omega t - kb) \sin k \Delta \} ds$$

$$dy = f \sin(\omega t - kb) (\cos k \Delta) ds - f \cos(\omega t - kb) (\sin k \Delta) ds.$$

The magnitude of the vibration at P due to wavefront between limits M and R_0 is

$$b = (a+b) \frac{s^2}{\lambda}$$

$$y_p = f \left[\sin(\omega t - kb) \int_0^s \cos(k\Delta) ds - \cos(\omega t - kb) \int_0^s \sin(k\Delta) \cdot ds \right]$$

writing the first integral as I_1 and the second integral as I_2 and setting

$$I_1 = A \cos \phi$$

$$I_2 = A \sin \phi$$

$$\text{so, } A \cos \phi = \int_0^s \cos(k\Delta) ds = \int_0^s \cos(k\Delta) ds$$

$$A \sin \phi = \int_0^s \sin(k\Delta) ds = \int_0^s \sin(k\Delta) ds$$

$$\text{so, } y_p = f \left[\sin(\omega t - kb) A \cos \phi - A \cos(\omega t - kb) \sin \phi \right]$$

$$y_p = f A \sin \{ \omega t - kb - \phi \}$$

$$\text{where } A^2 = I_1^2 + I_2^2$$

$$\tan \phi = \frac{I_2}{I_1}$$

Thus the integrals I_1 and I_2 are called Fresnel Integrals.

Now, from $\Delta O R P$

$$\begin{aligned} r^2 &= a^2 + (a+b)^2 - 2a(a+b) \cos \theta \\ &= (a+b)^2 + a^2 - 2a(a+b) \left[1 - \frac{\theta^2}{2!} \right] \quad \left\{ \begin{array}{l} \text{Since for small } \theta \\ \cos \theta \approx 1 - \frac{\theta^2}{2!} \end{array} \right. \\ &= b^2 + a(a+b) \theta^2 \end{aligned}$$

$$\text{But } \theta = \frac{s}{a}$$

$$\therefore r^2 = b^2 + \frac{a(a+b)s^2}{a^2}$$

$$\text{or, } (b+\Delta)^2 = b^2 \left[1 + \frac{(a+b)}{ab^2} s^2 \right]$$

$$\text{or, } b+\Delta = b \left[1 + \frac{(a+b)}{ab^2} s^2 \right]^{1/2}$$

$$\text{since } \frac{ab^2}{a+b} \gg s^2$$

$$\text{So, } b+\Delta = b \left[1 + \frac{1}{2} \frac{(a+b)}{ab^2} s^2 \right]$$

$$\text{so, } \Delta = \frac{(a+b)}{2ab} s^2$$

$$b + \Delta = b + \frac{a+b}{2ab} s^2$$

$$\Delta = \frac{a+b}{2ab} s^2$$

now, we introduce the relation

$$k \Delta = \frac{1}{2} \pi v^2$$

$$\text{or, } \frac{2\pi}{\lambda} \left\{ \frac{(a+b)}{2ab} s^2 \right\} = \frac{1}{2} \pi v^2$$

$$\text{or, } v = s \sqrt{\frac{2(a+b)}{ab\lambda}}$$

$$\text{so, } dv = ds \sqrt{\frac{2(a+b)}{ab\lambda}}$$

Hence
$$I_1 = \int_0^s \cos(k\Delta) ds = \sqrt{\frac{ab\lambda}{2(a+b)}} \int_0^v \cos\left(\frac{1}{2} \pi v^2\right) dv$$

$$I_2 = \int_0^s \sin(k\Delta) ds = \sqrt{\frac{ab\lambda}{2(a+b)}} \int_0^v \sin\left(\frac{1}{2} \pi v^2\right) dv$$

$$\text{So, } A^2 = I_1^2 + I_2^2$$

$$A^2 = B^2 \left\{ \int_0^v \cos\left(\frac{1}{2} \pi v^2\right) dv \right\}^2 + \left\{ \int_0^v \sin\left(\frac{1}{2} \pi v^2\right) dv \right\}^2$$

$$\text{where, } B = \sqrt{\frac{ab\lambda}{2(a+b)}}$$

Cornu's Spiral

The Fresnel's integrals are defined by the following equations

$$x = C(u) = \int_0^u \cos\left(\frac{1}{2} \pi v^2\right) dv$$

$$y = S(u) = \int_0^u \sin\left(\frac{1}{2} \pi v^2\right) dv$$

$$dx = \cos\left(\frac{\pi v^2}{2}\right) dv, \quad dy = \sin\left(\frac{\pi v^2}{2}\right) dv$$

$$\text{so, } \tan \psi = \frac{dy}{dx} = \frac{\sin\left(\frac{\pi v^2}{2}\right)}{\cos\left(\frac{\pi v^2}{2}\right)} = \tan\left(\frac{\pi v^2}{2}\right)$$

$$\text{or, } \tan \psi = \tan \left(\frac{\pi v^2}{2} \right)$$

$$\text{or, } \psi = \frac{\pi v^2}{2}$$

so when $v=0$, $\psi=0$

It means the curve is parallel to the x -axis at the origin.

→ The values of Fresnel integrals can be calculated for different values of v . The two integrals represent the vertical and horizontal components of the resultant amplitude and the intensity at any point is proportional to the square of the resultant amplitude.

$$I = f^2(x^2 + y^2)$$

→ The coordinates of Cornu's spiral are given by

$$x = \int_0^v \cos\left(\frac{1}{2} \pi v^2\right) dv$$

$$y = \int_0^v \sin\left(\frac{1}{2} \pi v^2\right) dv$$

The length element dv along the spiral is given by

$$dv = \sqrt{(dx)^2 + (dy)^2}$$

$$\text{Since } \psi = \frac{\pi v^2}{2}$$

$$\text{so, } \frac{d\psi}{dv} = \pi v$$

$$\text{or, } \frac{dv}{d\psi} = \frac{1}{\pi v}$$

Here $\frac{dv}{d\psi}$ measures the radius of curvature of the spiral at any point under consideration.

$$\text{Since } \frac{dv}{d\psi} \propto \frac{1}{v}$$

It shows that with the increase in the value of v , the radius of curvature of the curve gradually decreases and takes the ~~shape~~ shape of a spiral.

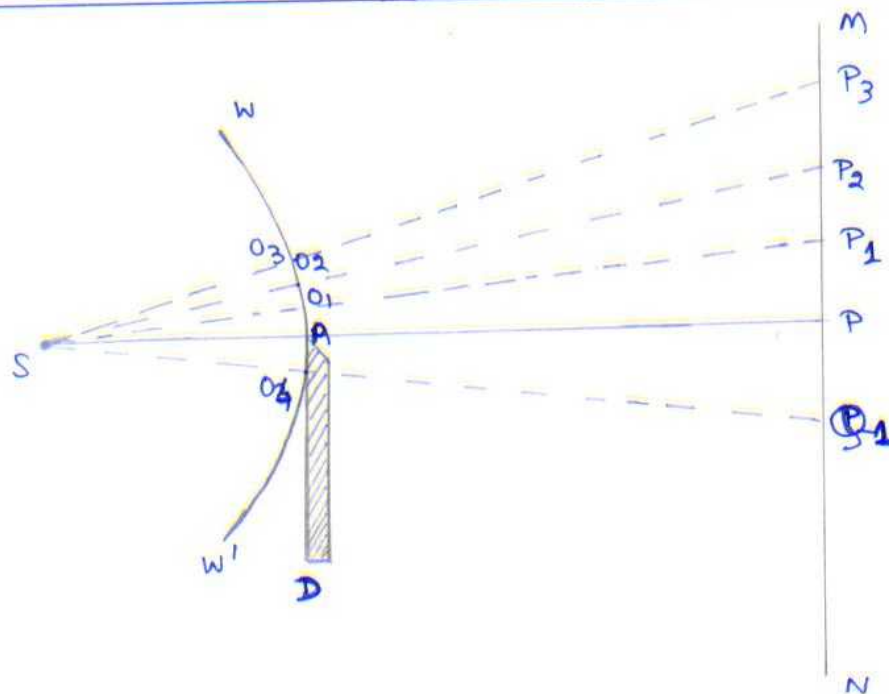
• Fresnel integrals have following

$$c(w) = \int_0^w \cos\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad s(w) = \int_0^w \sin\left(\frac{\pi t^2}{2}\right) dt$$

and $c(0) = s(0) = 0$

$$c(-t) = -c(t) \quad \text{and} \quad s(-t) = -s(t)$$

• Diffraction at a straight edge using Cornu's spiral

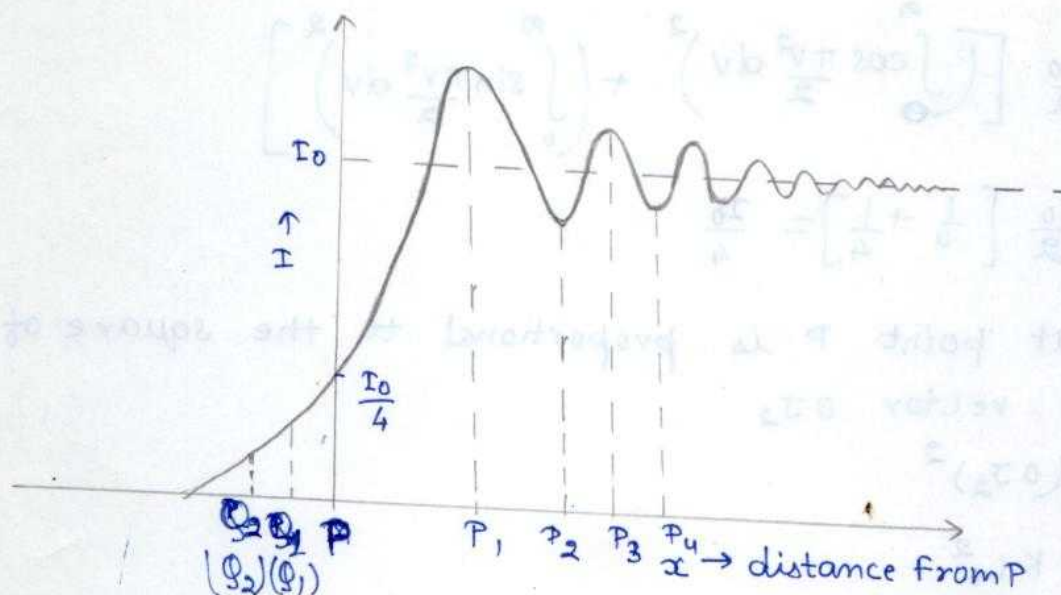


S is a narrow slit illuminated by monochromatic light of wavelength λ . The length of the slit is perpendicular to the plane of the paper. AD is a straight edge and the length of the edge is parallel to the length of the slit. WW' is the incident cylindrical wavefront.

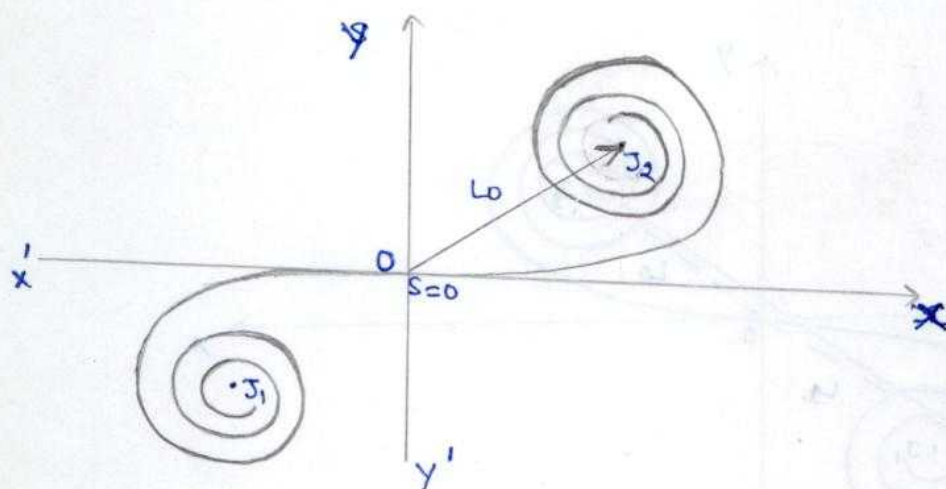
Cornu's spiral helps to obtain qualitatively distribution at a point on the screen.

Consider the points P, P_1, P_2 in the illuminated portion and point Q_1 in the geometrical shadow region of the screen MN. A, O_1, O_2, O_3 and O_4 are the poles of the wavefront for the points P, P_1, P_2, P_3 and Q_1 respectively.

Intensity pattern due to straight edge



Intensity at point P



The pole of the wave front is the origin of coordinates. The lower half of the wavefront is cut off by the obstacle (ie. straight edge) and the intensity at P is due to the upper half of the wavefront between A and W . The intensity is equal to $\frac{I_0}{4}$ where I_0 is the intensity due to whole wavefront.

$$I = K_1 [x^2 + y^2]$$

$$I_0 = K_1 \left\{ \left[\int_{-\infty}^{\infty} \cos \frac{\pi v^2}{2} dv \right]^2 + \left[\int_{-\infty}^{\infty} \sin \frac{\pi v^2}{2} dv \right]^2 \right\}$$

$$I = 2K_1$$

Let I_0 be the intensity due to the whole wavefront

$$I_0 = 2K_1$$

Thus $K_1 = \frac{I_0}{2}$

Now, at Point P

$$I = \frac{I_0}{2} \left[\left(\int_0^\infty \cos \frac{\pi v^2}{2} dv \right)^2 + \left(\int_0^\infty \sin \frac{\pi v^2}{2} dv \right)^2 \right]$$

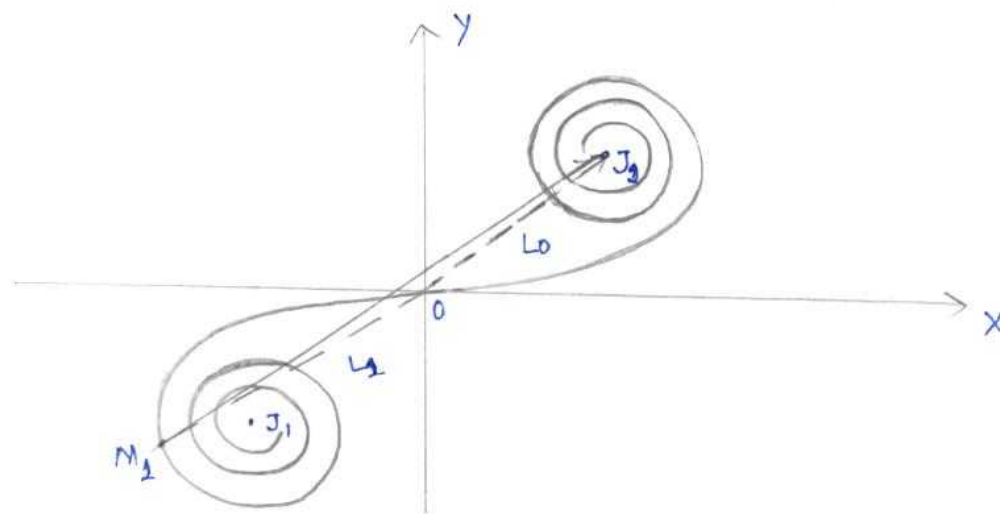
$$= \frac{I_0}{2} \left[\frac{1}{4} + \frac{1}{4} \right] = \frac{I_0}{4}$$

Also intensity at point P is proportional to the square of the amplitude vector OJ_2

$$I_P \propto (OJ_2)^2$$

$$I_P = K L_0^2$$

Intensity at point P_1



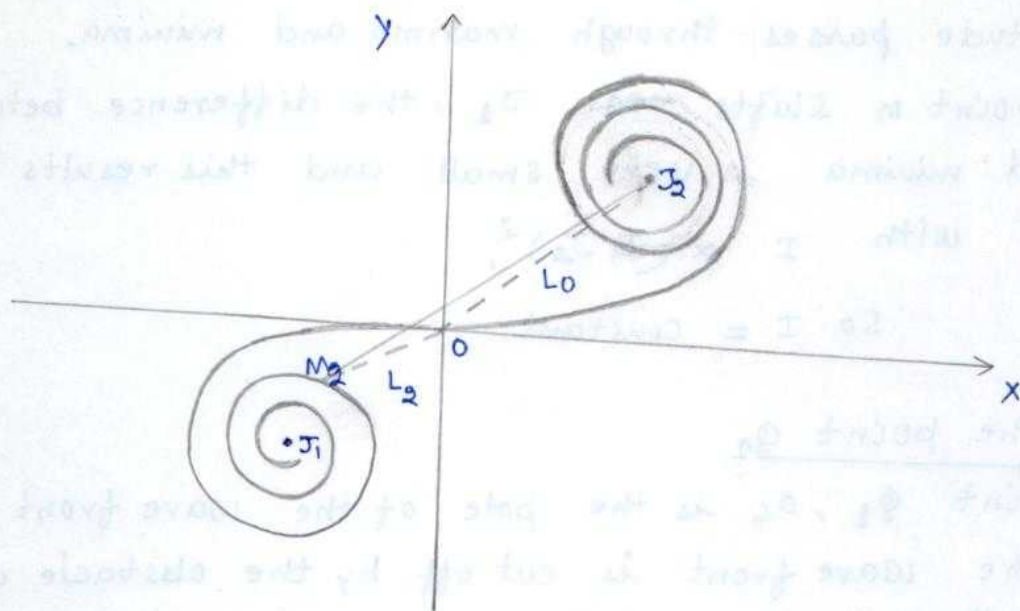
For point P_1 , O_1 is the pole of the wavefront, ~~and let the~~ Then the intensity at P_1 is due to the entire half wavefront above O_1 and also ^{due to} the exposed portion of the lower wavefront between A and O_1 . Let the exposed portion of the wavefront between A and O_1 , corresponds to the spiral OM_1 .

Then the intensity in this case is proportional to the square of the vector M_1J_2

$$I_{P_1} \propto (M_1J_2)^2$$

$$\propto (L_1^2 + L_0^2)$$

Intensity at point P_2



For the point P_2 , O_2 is the pole of the wavefront and length of the exposed portion of the wavefront between A and O_2 correspond to the length of the spiral OM_2 . The intensity at P_2 is proportional to the square of the amplitude vector $M_2 J_2$ which is a minimum.

$$I_{P_2} \propto (M_2 J_2)^2$$

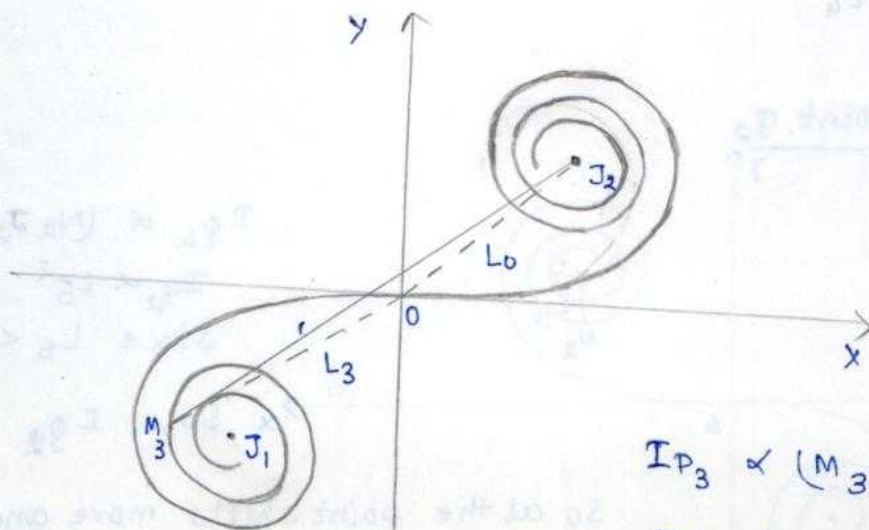
$$I_{P_2} \propto (L_0^2 + L_2^2)$$

Since $L_1 \neq L_2$

$$\text{so } I_{P_1} > I_{P_2}$$

so P_2 is a minimum intensity point.

Intensity at point P_3



$$I_{P_3} \propto (M_3 J_2)^2$$

$$I_{P_3} \propto (L_0 + L_3)^2$$

Since $L_3 > L_2$ so $I_{P_3} > I_{P_2}$

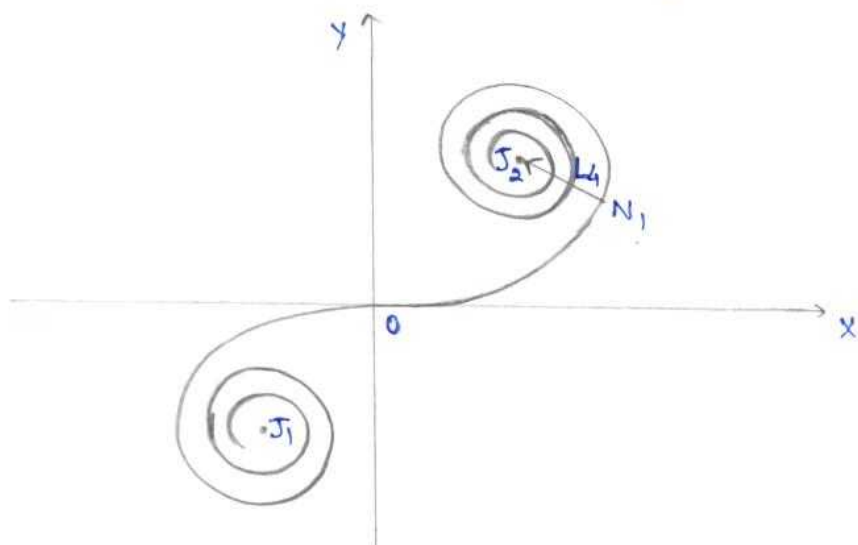
Thus, when the point P shifts away from P, the resultant intensity is proportional to the square of the amplitude vector whose magnitude passes through maxima and minima.

When the point M shifts near J_1 , the difference between the maxima and minima is very small and this results in uniform illumination. with $I \propto (J_1 J_2)^2$.

So $I = \text{Constant}$

Intensity at the point Q_1

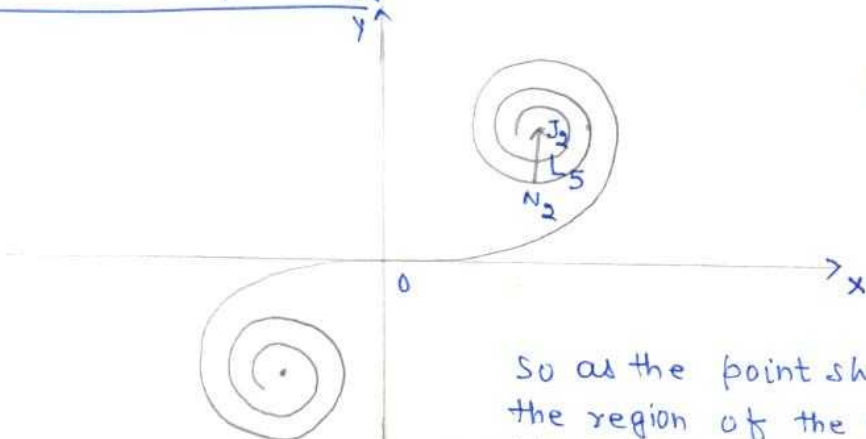
For the point Q_1 , O_4 is the pole of the wave front. The lower half of the wave front is cut off by the obstacle and the portion of the wavefront between A and O_4 is also obstructed corresponding to the length of the spiral ON_1 . The intensity at Q_1 is proportional to the amplitude $N_1 J_2$.



$$\text{So } I_{Q_1} \propto (N_1 J_2)^2$$

$$I_{Q_1} \propto L_4^2$$

Intensity at the point Q_2



$$I_{Q_2} \propto (N_2 J_2)^2$$

$$I_{Q_2} \propto L_5^2$$

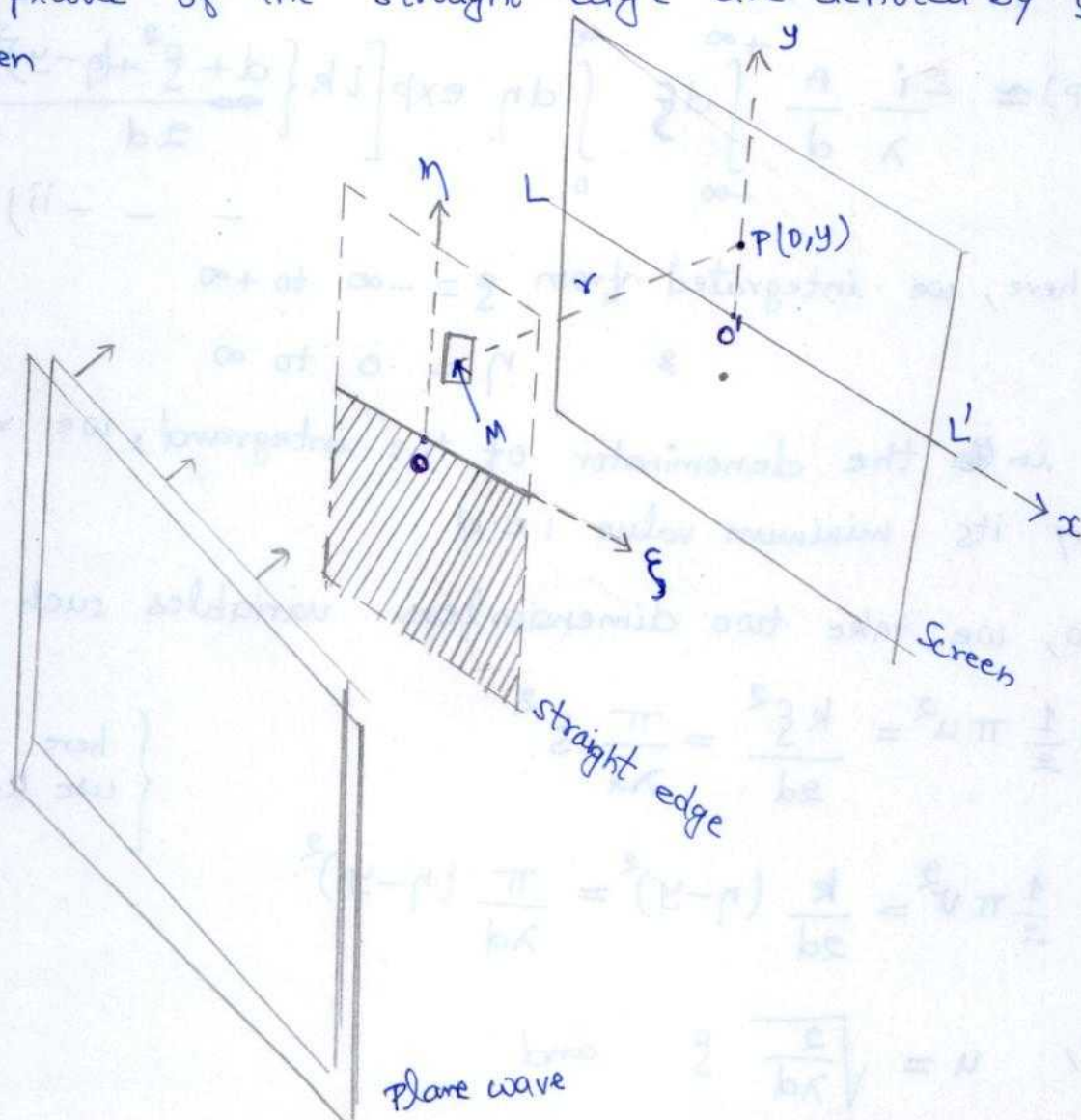
Since $L_5 < L_4$

So, $I_{Q_2} < I_{Q_1}$.

So as the point shifts more and more into the region of the geometrical shadow, the point N shifts more and more towards J_2 . Thus, the magnitude of $N J_2$ gradually decreases and intensity falls off gradually.

Straight Edge Diffraction pattern $\Rightarrow$

Consider a plane wave incident normally on a straight edge. From the geometry of arrangement, it can be seen that ^{on the screen} there will be no intensity variation along the direction $\hat{a}$ parallel to the length of the straight edge, i.e. along the x -axis. Therefore without the loss of generality, we may assume the coordinates of any point P (on the screen) to be $(0, y)$; where the origin has been assumed to be on the edge of the geometrical shadow. If the x and y coordinates of an arbitrary point M on the plane of the straight edge are denoted by ξ and η , then



$$r = MP = \sqrt{\xi^2 + (\eta - y)^2 + d^2}$$

$$\approx d + \frac{\xi^2 + (\eta - y)^2}{2d}$$

where d = distance betⁿ the straight edge and the screen

The field at the point P due to waves emanating from the area $d\xi d\eta$ (around the point M) will be proportional to

$$\frac{Ae^{ikr}}{r} d\xi d\eta \quad \text{--- i)}$$

where, $r = MP$

Total field at point P , will be then $u(P)$, given by

$$u(P) \approx -\frac{i}{\lambda} \frac{A}{d} \int_{-\infty}^{+\infty} d\xi \int_0^{\infty} d\eta \exp\left[ik \left\{ \frac{d + \xi^2 + (\eta - y)^2}{2d} \right\}\right] \quad \text{--- ii)}$$

here, we integrated from $\xi = -\infty$ to $+\infty$

& $\eta = 0$ to ∞

also in the denominator of the integrand, we replaced r by its minimum value i.e. d

Now, we take two dimensionless variables such that

$$\frac{1}{2} \pi u^2 = \frac{k \xi^2}{2d} = \frac{\pi}{\lambda d} \xi^2$$

here we use $k = \frac{2\pi}{\lambda}$

and $\frac{1}{2} \pi v^2 = \frac{k (\eta - y)^2}{2d} = \frac{\pi}{\lambda d} (\eta - y)^2$

so, $u = \sqrt{\frac{2}{\lambda d}} \xi$ and

$$v = \sqrt{\frac{2}{\lambda d}} (\eta - y)$$

Substituting in eqn ii), it becomes

$$u(P) = -\frac{i}{2} u_0 \int_{-\infty}^{+\infty} \exp\left(\frac{i\pi u^2}{2}\right) du \int_{v_0}^{\infty} \exp\left(\frac{i\pi v^2}{2}\right) dv \quad \text{--- iii)}$$

where, $v_0 = -\sqrt{\frac{2}{\lambda d}} y$

and $u_0 = A e^{ikd}$ represents the field at point P in the absence of straight edge.

Since,

$$\int_{-\infty}^{+\infty} \exp\left[\frac{i\pi u^2}{2}\right] du = (1+i)$$

and

$$\int_{v_0}^{\infty} \exp\left(\frac{i\pi v^2}{2}\right) dv = \left[\int_0^{\infty} \cos\left(\frac{\pi}{2} v^2\right) dv - \int_0^{v_0} \cos\left(\frac{\pi}{2} v^2\right) dv \right] + i \left[\int_0^{\infty} \sin\left(\frac{\pi}{2} v^2\right) dv - \int_0^{v_0} \sin\left(\frac{\pi}{2} v^2\right) dv \right]$$

$$= \left[\frac{1}{2} - C(v_0) \right] + i \left[\frac{1}{2} - S(v_0) \right]$$

where $C(v_0)$ and $S(v_0)$ are fresnel integrals

So, putting these in eqn iii) we get

$$u(P) = \left(\frac{1-i}{2}\right) u_0 \left[\left\{ \frac{1}{2} - C(v_0) \right\} + i \left\{ \frac{1}{2} - S(v_0) \right\} \right] \quad \text{--- iv)}$$

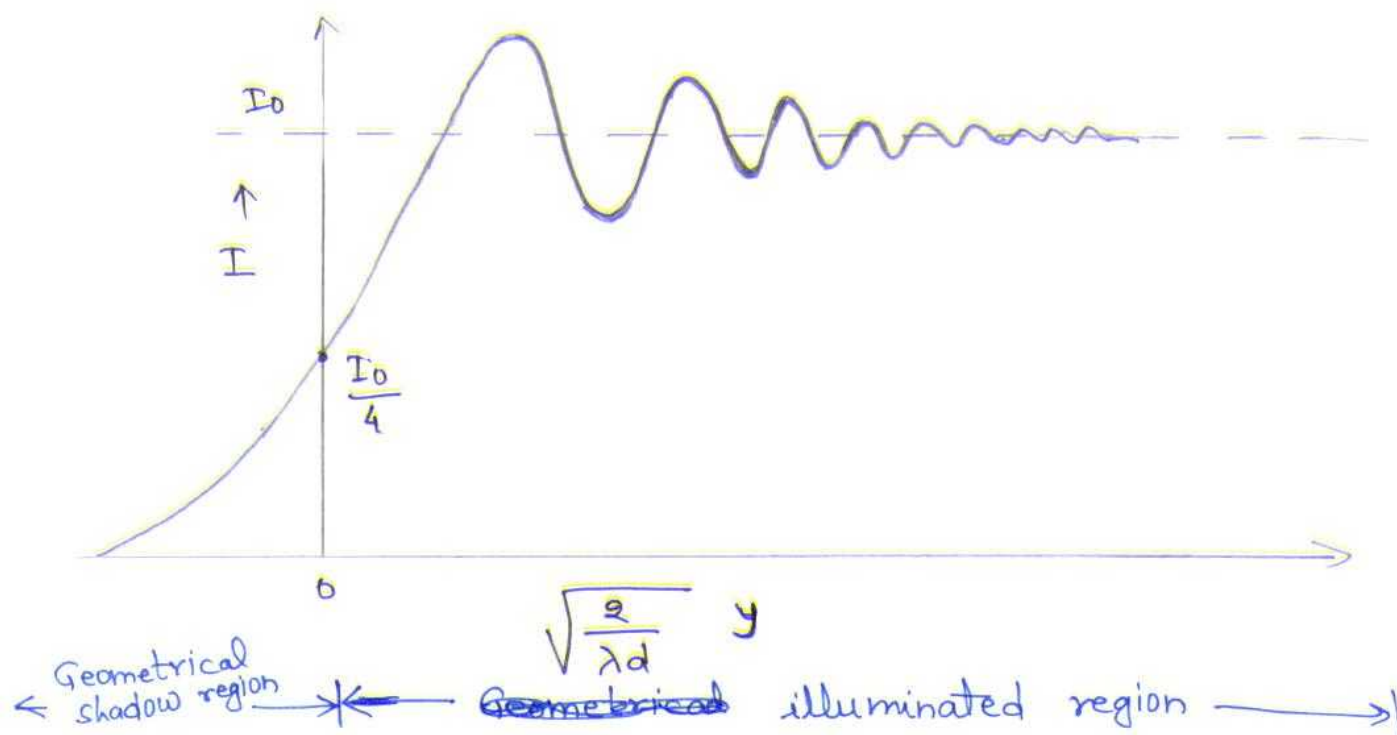
So, the intensity distribution would be

$$I(P) = \frac{I_0}{2} \left[\left\{ \frac{1}{2} - C(v_0) \right\}^2 + \left\{ \frac{1}{2} - S(v_0) \right\}^2 \right]$$

where, $C(v_0)$ and $S(v_0)$ are fresnel integrals.

also, $v_0 = -\sqrt{\frac{2}{\lambda d}} y$

The intensity pattern of the straight edge has been shown to be



For points in the illuminated region, the value of v_0 becomes negative and the corresponding point P lies in the third quadrant. Let P be the point corresponding to $\tau = v_0$, and C corresponding to $\tau = \infty$

$$\therefore \left[\frac{1}{2} - C(v_0) \right] + i \left[\frac{1}{2} - S(v_0) \right] = [C(\infty) - C(v_0)] + i [S(\infty) - S(v_0)]$$

$$= (PC) e^{i\phi}$$

$\therefore$ eqn iv) becomes

$$u(P) = \frac{(1-i)(PC) e^{i\phi}}{2} u_0$$

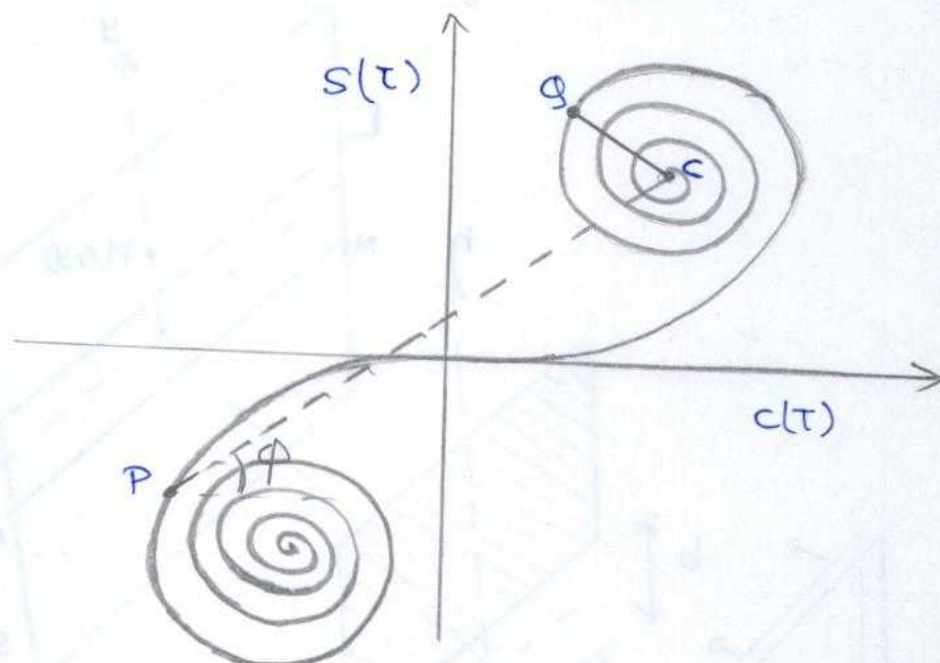
$$\Rightarrow \boxed{I(P) = \frac{1}{4} (PC)^2 I_0}$$

- For geometrical shadow region $v_0 = (+)ve$ so the corresponding point Q lies in the first quadrant. Let Q be the point corresponding to $\tau = \infty$.

$$\therefore \left[\frac{1}{2} - c(u_0) \right] + i \left[\frac{1}{2} - s(u_0) \right] = [c(\infty) - c(u_0)] + i[s(\infty) - s(u_0)] \\ = (gc) e^{i\psi}$$

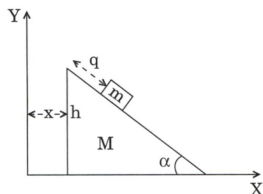
$$\therefore u(g) = \frac{(1-i)(gc)}{2} e^{i\psi} u_0$$

$$\Rightarrow \boxed{I(g) = \frac{1}{4} (gc)^2 I_0}$$



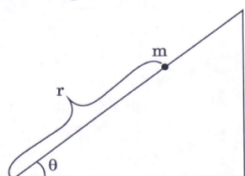
(Cornu's spiral)

- (c) A block of mass m is sliding on a wedge of mass M as shown in the figure below. The wedge can slide on the horizontal table. Find the equation of motion. 15



- (c) Derive Euler-Lagrange equations of motion from Hamilton's principle for any bilateral holonomic system having no non-potential forces and n degree of freedom. Can it be applied to non-holonomic systems? 15

- A particle of mass m rests on a smooth plane. The plane is then raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane, as shown in the figure below.



- Write down the Lagrangian and determine the equations of motion. Solve the resulting equation for " r " to obtain the expression for $r(t)$. 20

1:45 AM Thu 12 Nov Not Secure edshare.soton.ac.uk

Question

A particle of mass m rests on a smooth plane. The plane is raised to an incline ϕ at a constant rate α ($\phi = 0$ at $t = 0$), causing the particle to move down the plane. Express the Lagrangian in polar coordinates of the particle in a coordinate system whose origin is at the foot of the plane. Hence determine the motion of the particle.

Answer

$\phi = \alpha t$

$L = K.E. - P.E. = \frac{1}{2}m(\dot{r}^2 - r^2\dot{\phi}^2) - mgr \sin \alpha$

$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \Rightarrow \ddot{r} - \alpha^2 r = -g \sin \alpha$

This has solution $r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$

Initially $r(0) = r_0$, $\dot{r}(0) = 0$ whence we can find A and B

Thus $r(t) = \frac{1}{2} \left[r_0 - \frac{g}{2\alpha^2} \right] e^{\alpha t} + \frac{1}{2} \left[r_0 + \frac{g}{2\alpha^2} \right] e^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$

- (a) Construct the Hamiltonian of a charged particle with charge q and mass m moving with the velocity $\vec{v}$ in the external electromagnetic field, $\vec{E} = E_0 \hat{i}$, $\vec{B} = B_0 \hat{k}$, where E_0 and B_0 are constants. 8

- (a) A particle of mass m is constrained to move on a plane curve $xy = c$ with $c > 0$, under gravity, where y -axis is vertical. Construct the Lagrangian of the system and obtain the Euler-Lagrange equation of motion. 10

- (a) In special theory of relativity, the Hamiltonian of a free particle with rest mass m_0 is given by $\sqrt{p^2 c^2 + m_0^2 c^4}$. Obtain the Lagrangian of the system using Legendre transformation. 10

- (a) A bead slides on a wire in the shape of a cycloid described by the equations

$$x = a(\theta - \sin \theta)$$

$$y = a(1 + \cos \theta) \text{ with } 0 \leq \theta \leq 2\pi.$$

Find the Lagrangian and equation of motion. 8

- (b) Show that the relativistic invariance laws of conservation of momentum lead to the concepts of variation of mass with velocity and mass energy equivalence. 8

$$xy = c$$

$$x = \frac{c}{y}$$

$$L = T - V$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} \dot{y} = \frac{1}{2} m \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right)$$

$$= \frac{1}{2} m \left(\left(\frac{c}{y^2} \right)^2 + 1 \right) \dot{y}^2$$

$$V = +mgy$$

$$L = \frac{1}{2} m \dot{y}^2 \left(\frac{c}{y^2} + 1 \right) - mgy$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt} \left[m \dot{y} \left(\frac{c}{y^2} + 1 \right) \right] - \left[\frac{1}{2} m \dot{y}^2 \left(-\frac{2c}{y^3} \right) - mg \right] = 0$$

$$m \dot{y} \left(\frac{c}{y^2} + 1 \right) + \frac{2mc}{y^3} \dot{y}^2 + mg = 0$$

$$\ddot{y} \left(\frac{c}{y^2} + 1 \right) + \frac{2c}{y^3} \dot{y}^2 + g = 0$$

$$\ddot{y} + \frac{2c}{y^3} \dot{y}^2 + \frac{g}{1 + \frac{c}{y^2}} = 0$$

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

$$H = \sum p \dot{q} - L$$

(i) State and prove Hamilton's principle and use it to prove that the shortest distance between two points in space is a straight line joining them.

(ii) Use Hamiltonian mechanics to find the differential equation for planetary motion, moving under force $f(r) = -\frac{k}{r^2}$ and prove that the areal velocity is constant.

8+7=15

What are cyclic coordinates for a system of particles? Discuss the significance of the Hamilton's principle.

5+5

Show how Lagrange's equations can be derived from Hamilton's principle.

15

The Lagrangian for a particle has the form

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - U(r)$$

Obtain its form in plane polar coordinates and then get expression for the Hamiltonian using appropriate transformation.

Obtain the equation of motion of a simple pendulum using Lagrangian formalism. Hence, obtain expression for the angular frequency of a simple harmonic oscillator.

15

Q. 1(a) Using the concept of D'Alembert's principle, show that the generalized force can be defined as

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

where $\vec{r}_i$ is the Cartesian coordinate of the i^{th} particle experiencing external force $\vec{F}_i$ and q_j stands for the generalized coordinate. Discuss the significance of the above expression.

8

1(a) State Hamilton's principle for a system of particles. If I and L represent the action integral and the Lagrangian function, respectively, write down the mathematical form of Hamilton's principle and explain clearly the significance of the same.

15

(a) Express Lagrange's equation of motion for the cyclic coordinate q_j and show that the result leads to the general conservation theorem for the generalized momentum coordinates.

(a) Starting with Newton's second law of motion, establish D'Alembert's principle and discuss its significance.

8

(b) Using D'Alembert's principle, show that the following relation can be obtained for a system of particles under generalized coordinates

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_j Q_j \delta q_j$$

$$\text{with } Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

What is the significance of Q_j ?

16

(c) State Hamilton's principle for the motion of a monogenic system.

Use the calculus of variation to deduce Lagrange's equation that follows from Hamilton's principle. Explain carefully all the terms used in the derivation.

16

① D.O.F = Coordinates - ind. constraints

② Holonomic: $f(r_1, r_2, \dots, t) = 0 \rightarrow$ Also has superfluous coordinates
 Non holonomic [$\uparrow$ can't be expressed this way]

③ Rheonomic (time explicit)
 Scleronomous (time not explicitly included)

④ Generalised Coordinates: $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n, t)$

⑤ Virtual work: $\sum \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$ (at equilibrium)

⑥ D'Alembert's Principle
 $\sum (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$

⑦ Lagrange's Equations

7.1 From D'Alembert: [use: $\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$; $\frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_i}{\partial \dot{q}_k} \right) = \frac{\partial \mathbf{r}_i}{\partial \dot{q}_k}$]

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} - G_k = 0 \quad \text{where } G_k = \sum \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_k} = \text{Generalised force}$$

For a conservative system:

(i) $G_k = -\frac{\partial V}{\partial q_k}$ & $\frac{\partial V}{\partial \dot{q}_k} = 0$

(ii) $L = T - V$ (Lagrangian)

(iii) $\left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \right\}$ (Lagrange's eqⁿ) [n equations]

• For polar: $L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$

• For EM forces, $V = q(\phi - \vec{v} \cdot \vec{A})$ (generalised potential)

7.2 From Hamiltonian principle

• Principle: for conservative holonomic sys., motion is such that action integral has stationary or extremum value for path of motion

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (\text{where } L = \text{Lagrangian})$$

Let: $L = L(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t)$

$$\delta L = \sum \frac{\partial L}{\partial q_k} \delta q_k + \sum \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k$$

$$0 = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left[\sum \frac{\partial L}{\partial q_k} \delta q_k + \sum \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt$$

$$(\delta \dot{q}_k = \frac{d}{dt}(\delta q_k))$$

Use integration by parts

$$\int \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt = \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt$$

At end of path, coordinates are definite $\Rightarrow \delta q_k(t_2) = \delta q_k(t_1) = 0$

$$\Rightarrow \int_{t_1}^{t_2} \left[\sum \frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt = 0$$

$$\Rightarrow \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \right] \quad \frac{\partial f}{\partial y} - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0$$

• Gauge variation:

$$L' = L + \frac{dF}{dt} \quad \text{where: } F = F(q_1, q_2, \dots, q_n, t)$$

Hamiltonian Dynamics

① $p = \frac{\partial T}{\partial \dot{x}} \Rightarrow \boxed{p_k = \frac{\partial L}{\partial \dot{q}_k}}$ conjugate momentum
(if V is function of velocity)

use in LB: $\frac{d}{dt} p_k - \frac{\partial L}{\partial q_k} \Rightarrow \dot{p}_k = \frac{\partial L}{\partial q_k}$

If q_k does not appear in $L \Rightarrow \frac{\partial L}{\partial q_k} = 0 \Rightarrow \boxed{p_k = \text{constant} = \frac{\partial L}{\partial \dot{q}_k}}$

Cyclic or ignorable variable
(use to get $m\ddot{r} = \text{const.}$ in the central force motion)

② Jacobi's Integral:

• Let $L(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t)$

• If $\frac{\partial L}{\partial t} = 0$ (L does not explicitly depend on t) then, the first integral / Jacobi Integral is

a constant & equal to total energy

$$h(q_1, q_2, \dots, \dot{q}_1, \dot{q}_2, \dots, t) = \sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{constant}$$

③ Hamiltonian ($H(p_k, q_k, t)$)

$$\frac{d}{dt} (\sum p_k \dot{q}_k - L) = -\frac{\partial L}{\partial t} \Rightarrow \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\boxed{H = \sum p_k \dot{q}_k - L} \quad \text{This is Legendre transform}$$

Now, if t isn't explicit then H is constant & equal to total energy

④ Hamilton's eqⁿ:

Let $H = H(q_k, p_k, t)$

$$dH = \sum \frac{\partial H}{\partial q_k} dq_k + \sum \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \quad -①$$

By Euler rule for homogenous system

$$\sum \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T \quad (\text{for 2 degree})$$

$$\Rightarrow \text{Also, } H = \sum p_k \dot{q}_k - L$$

$$dH = \sum \dot{q}_k dp_k + \sum p_k d\dot{q}_k - dL$$

$$\left(dL = \sum \frac{\partial L}{\partial q_k} dq_k + \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt \right)$$

$$\dot{q}_k = \frac{\partial L}{\partial p_k} \quad p_k = \frac{\partial L}{\partial \dot{q}_k}$$

$$\Rightarrow dL = \sum \dot{q}_k dq_k + \sum p_k d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow dH = \sum \dot{q}_k dp_k - \sum p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt \quad -②$$

Compare ① & ②

$$\boxed{\frac{\partial H}{\partial p_k} = \dot{q}_k} \quad \boxed{\frac{\partial H}{\partial q_k} = -\dot{p}_k} \quad \boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

2n first order Hamilton's equations

Change L to H by Legendre transform

$$L = L(q, \dot{q}, t) \longleftrightarrow H = H(q, p, t)$$

$$\text{new } f^n \leftarrow g = f - dx \xrightarrow{\text{old } f^n} \text{change}$$

replace by

$$\Rightarrow g = -(f - Ex)$$

$$H = -L - \sum p_i \dot{q}_i$$

$$H = \sum p_i \dot{q}_i - \sum L(q_i, \dot{q}_i, t)$$

$$dH = \sum p_i d\dot{q}_i + \sum \dot{q}_i dp_i - \sum \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \right)$$

$$\text{Let } p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (\text{generalised momentum})$$

$$dH = \sum p_i d\dot{q}_i + \sum \dot{q}_i dp_i - \sum \left(\frac{\partial L}{\partial q_i} dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \right)$$

$$dH = \sum \dot{q}_i dp_i - \sum \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt$$

$$\text{But } \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

$$\Rightarrow \frac{\partial H}{\partial t} = \frac{\partial L}{\partial q_i} \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}$$

$$\Rightarrow dH = \sum \dot{q}_i dp_i - \sum p_i dq_i - \frac{\partial L}{\partial t} dt$$

$$\text{Also, } dH = \sum \frac{\partial H}{\partial q_i} dq_i + \sum \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

(compare & get Hamiltonian eqⁿ)

Euler Lagrange eqⁿ:

$$\left[\frac{\partial f}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) = 0 \right]$$

where: $y'_k = \frac{dy_k}{dx}$; $f = f(x, y, y')$

Now compare with

$$L(x_k, \dot{x}_k, t)$$

$$\equiv x = t, y = x_k, f = L;$$

$$y' = \dot{x}_k$$

$$\delta \int_{x_1}^{x_2} f(y, y', x) dx = 0$$

5.2. THE CALCULUS OF VARIATIONS AND EULER-LAGRANGE'S EQUATIONS

Let us have a function $f(y, y', x)$ defined on a curve given by

$$y = y(x) \quad \dots(1)$$

between two points $A(x_1, y_1)$ and $B(x_2, y_2)$. Here, $y' = dy/dx$. We are interested in finding a particular curve $y(x)$ for which the line integral I of the function f between the two points

$$I = \int_{x_1}^{x_2} f(y, y', x) dx \quad \dots(2)$$

has a stationary value.

Suppose that APB be the curve for which I is stationary. Now, consider a neighbouring curve $AP'B$ with the same end points A and B . The point $P(x, y)$ of the curve APB corresponds to the point $P'(x, y + \delta y)$ of the curve $AP'B$, keeping x -coordinate of the points fixed. This defines a δ -variation of the curve. The variation is arbitrary but small and may be expressed as

$$\delta y = \frac{\partial y}{\partial \alpha} \delta \alpha = \eta(x) \delta \alpha \quad \dots(3)$$

where α is a parameter (independent of x) common to all points of the path and $\eta(x)$ is a function of x with the condition that

$$\delta y_1 = \delta y_2 = \eta(x_1) = \eta(x_2) = 0 \quad \dots(4)$$

By choosing different $\eta(x)$, we may construct different varied paths.

The corresponding variation in y' is

$$\delta y' = \eta'(x) \delta \alpha \quad \dots(5)$$

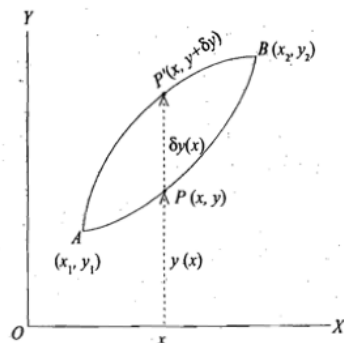


Fig. 5.1 : δ -variation

Variational Principles

139

Now, the integral on the varied path is

$$I' = \int_{x_1}^{x_2} f(y + \delta y, y' + \delta y', x) dx$$

$$\text{or} \quad I' = \int_{x_1}^{x_2} f(y + \eta \delta \alpha, y' + \eta' \delta \alpha, x) dx \quad \dots(6)$$

Since the variation is small, the integral I' may be obtained by considering only first order terms in the Taylor expansion of the function f i.e.,

$$I' = \int_{x_1}^{x_2} [f(y, y', x) + \frac{\partial f}{\partial y} \eta \delta \alpha + \frac{\partial f}{\partial y'} \eta' \delta \alpha] dx \quad \dots(7)$$

$$\text{Hence} \quad \delta I = I' - I = \delta \alpha \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' \right) dx \quad \dots(8)$$

$$\text{But} \quad \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta' dx = \left[\frac{\partial f}{\partial y'} \eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta dx$$

$$\text{or} \quad \int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta' dx = - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \eta dx \quad [\text{as } \eta(x_1) = \eta(x_2) = 0]$$

$$\text{Therefore,} \quad \delta I = \delta \alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta dx \quad \dots(9)$$

The condition that the integral I is stationary means that $\delta I = 0$, i.e.,

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] \eta dx = 0 \quad \dots(10)$$

As η is arbitrary, the integrand of (10) must be zero, i.e.,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0 \quad \dots(11)$$

which is known as **Euler-Lagrange equation**.

The result can easily be generalized to the case where f is a function of many independent variables y_k and their derivatives y'_k . However, y_k and y'_k are function of x . Then

$$\delta I = \delta \int_{x_1}^{x_2} f(y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n, x) dx = 0 \quad \dots(12)$$

leads to the Euler-Lagrange equations

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) = 0 \quad \dots(13)$$

where, $k = 1, 2, \dots, n$.

It is to be pointed out that in most of the problems the stationary value of the integral is seen to be a minimum but occasionally maximum.

140

Classical Mechanics

Hamilton's principle and Lagrange's equations : If we identify the Lagrangian to

$$L = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) \quad \dots(14)$$

to the function

$$f = f(y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n, x)$$

with the transformations

$$x \rightarrow t, y_k \rightarrow q_k, y'_k \rightarrow \dot{q}_k,$$

then the integral $I \rightarrow S$ defines the action integral

$$S = \int_{x_1}^{x_2} L dt \quad \dots(15)$$

This integral is known as **Hamilton's principal function**.

The variation

$$\delta S = \delta \int L dt = 0 \quad \dots(16)$$

corresponding to eq. (12) means that **the motion of the system from time $t = t_1$ to time $t = t_2$ is such that the line integral (15) has a stationary value for the correct path of the motion**. This is what is known as **Hamilton's principle**. The necessary condition for the Hamilton's principle ($\delta S = 0$) is given by Lagrange's equations of motion in place of Euler-Lagrange's equations, i.e.,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad \dots(17)$$

where q_k are the generalized coordinates.

These coordinates are independent corresponding to the independent variables y_k and hence the constraints are holonomic.

6.2. LEGENDRE TRANSFORMATIONS

This is a mathematical technique used to change the basis from one set of coordinates to another. If $f(x, y)$ is a function of two variables x and y , then the differential of this function can be written as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \text{ or } df = u dx + v dy \quad \dots(5)$$

where $u = \partial f / \partial x$ and $v = \partial f / \partial y$... (6)

Now, we want to change the basis from (x, y) to (u, v) so that u is now an independent variable and x is a dependent one. Let f' be a function of u and y such that

$$f' = f - ux \quad \dots(7)$$

Then, $df' = df - u dx - x du$

Substituting for df from (5), we get

$$df' = u dx + v dy - u dx - x du$$

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

$$f = f(x, y)$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Now to change from (x, y) to $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

$$f' = f - ux$$

$$df' = df - u dx - x du$$

$$= u dx + v dy - u dx - x du$$

$$df' = v dy - x du$$

or $df' = v dy - x du$... (8)

But f' is a function of u and y , therefore

$$df' = \frac{\partial f'}{\partial u} du + \frac{\partial f'}{\partial y} dy \quad \dots(9)$$

Comparing eqs. (8) and (9), we get

$$x = -\frac{\partial f'}{\partial u} \text{ and } v = \frac{\partial f'}{\partial y} \quad \dots(10)$$

These are the necessary relations for Legendre transformations.

The Lagrange bracket of two dynamical variables $F(q_k, p_k)$ and $G(q_k, p_k)$ is defined as

$$\{F, G\} = \sum_k \left[\frac{\partial q_k}{\partial F} \frac{\partial p_k}{\partial G} - \frac{\partial p_k}{\partial F} \frac{\partial q_k}{\partial G} \right] \quad \dots(28)$$

$$H = \sqrt{p^2 c^2 + m_0^2 c^4}$$

we know

$$\frac{\partial H}{\partial p_k} = \dot{q}_k \text{ \& } \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

$$\Rightarrow \dot{x} = \frac{pc^2}{\sqrt{p^2 c^2 + m_0^2 c^4}}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{pc^2}{\sqrt{p^2 c^2 + m_0^2 c^4}}$$

$$(p^2 c^2 + m_0^2 c^4) \dot{x}^2 = p^2 c^4$$

$$p^2 (c^4 - c^2 \dot{x}^2) = m_0^2 c^4 \dot{x}^2$$

$$\text{Now: } H = p\dot{x} - L$$

$$L = H + p\dot{x}$$

$$p^2 = \frac{m_0^2 c^4 \dot{x}^2}{(c^4 - c^2 \dot{x}^2)} = \frac{m_0^2 \dot{x}^2}{(1 - \dot{x}^2/c^2)}$$

$$\begin{aligned}
 & \sqrt{\frac{m_0^2 \dot{x}^2 c^2 + m_0^2 c^4}{1 - \dot{x}^2/c^2} + \frac{m_0^2 \dot{x}^2}{1 - \dot{x}^2/c^2}} \\
 &= \sqrt{\frac{m_0^2 \dot{x}^2 c^2 + m_0^2 c^4 - m_0^2 \dot{x}^2 c^2}{1 - \dot{x}^2/c^2}} \\
 &= \frac{m_0 c^2}{\sqrt{1 - \dot{x}^2/c^2}} + \frac{m_0 \dot{x}^2}{\sqrt{1 - \dot{x}^2/c^2}}
 \end{aligned}$$

$$L = \frac{m_0(c^2 + \dot{x}^2)}{\sqrt{1 - \dot{x}^2/c^2}}$$

① Lagrangian From Hamiltonian

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta L dt = 0 \quad L = L(q, \dot{q})$$

$$\Rightarrow \int_{t_1}^{t_2} \frac{\partial L}{\partial q_k} \delta q_k + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k$$

$$\Rightarrow \int_{t_1}^{t_2} \frac{\partial L}{\partial q_k} \delta q_k + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\delta q_k)$$

$$\Rightarrow \int_{t_1}^{t_2} \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} (\delta q_k) - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k$$

$$\Rightarrow \left[\frac{\partial L}{\partial t} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] = 0$$

② Hamilton from D'Alembert

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

→ extremum
→ action integral

Condition: $\delta t = 0$ at end points

• $\delta q = 0$ at end points

Now, $\sum (F_i - \dot{p}_i) \delta x_i = 0$ (D'Alembert)

$$\Rightarrow \sum F_i \delta x_i = \sum \dot{p}_i \delta x_i$$

For conservative: $F_i = -\nabla V \Rightarrow F_i = -\frac{\partial V}{\partial x_i}$

$$\sum -\frac{\partial V}{\partial x_i} \delta x_i = \sum \dot{p}_i \delta x_i$$

$$\Rightarrow -\delta V = \sum \dot{p}_i \delta x_i = \sum m_i \dot{v}_i \delta x_i$$

$$-\delta V = \sum m_i \ddot{x}_i \delta x_i =$$

$$\text{Let: } \sum m_i \dot{v}_i \delta x_i = \sum m_i \frac{d}{dt} (\dot{x}_i \delta x_i) - \sum m_i \dot{x}_i \frac{d}{dt} (\delta x_i)$$

$$\Rightarrow -\delta V = \sum m_i \frac{d}{dt} (\dot{x}_i \delta x_i) - \sum m_i \dot{x}_i \frac{d}{dt} (\delta x_i)$$

$$-\delta V = \frac{d}{dt} \sum m_i \dot{x}_i \delta x_i - \sum m_i \dot{x}_i \frac{d}{dt} (\delta x_i)$$

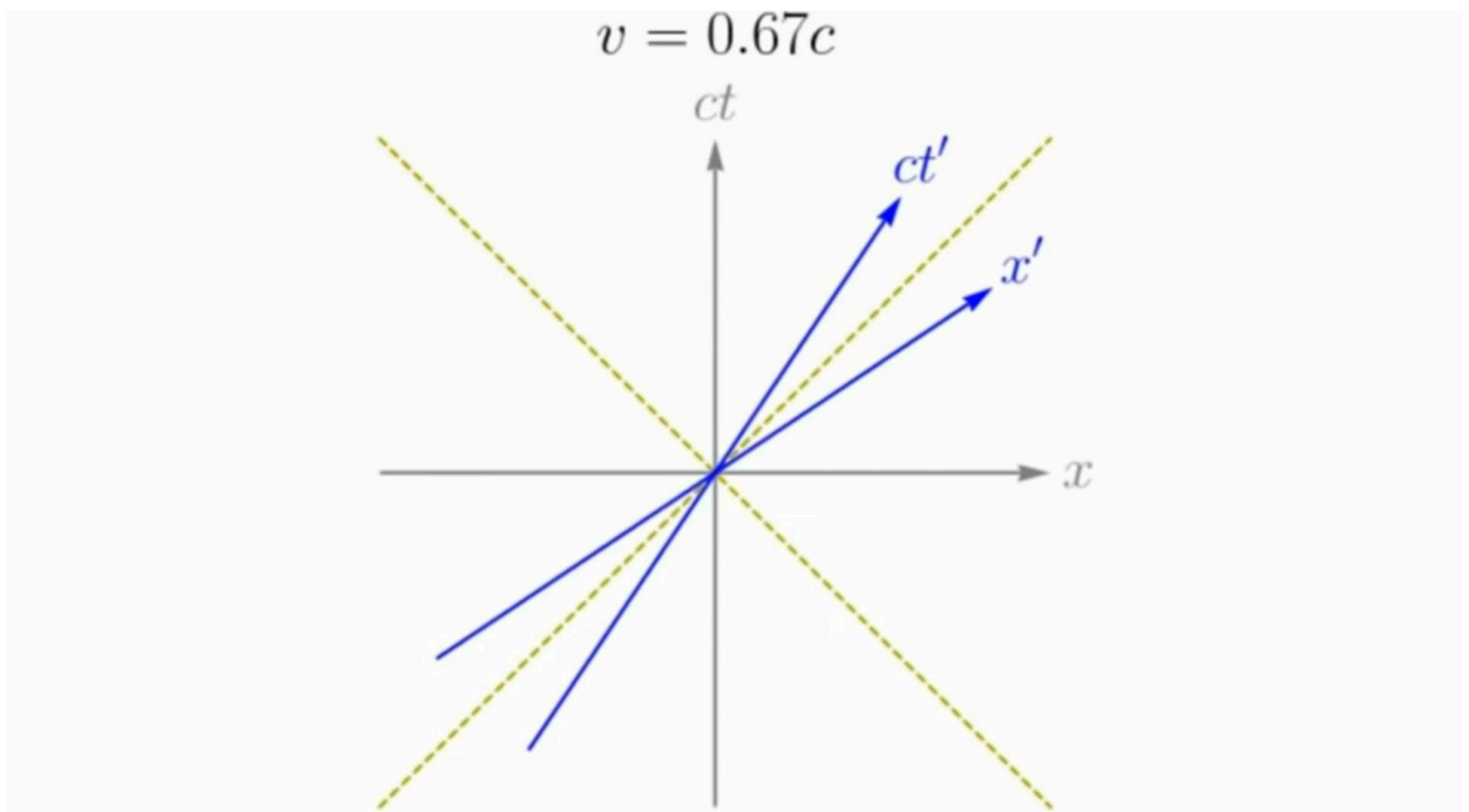
$$-\delta V = \frac{d}{dt} \sum m_i \dot{x}_i \delta x_i - \delta \left(\sum \frac{1}{2} m_i \dot{x}_i^2 \right)$$

$$\Rightarrow \delta T - \delta V = \frac{d}{dt} \sum m_i \dot{x}_i \delta x_i \quad \rightarrow \delta T$$

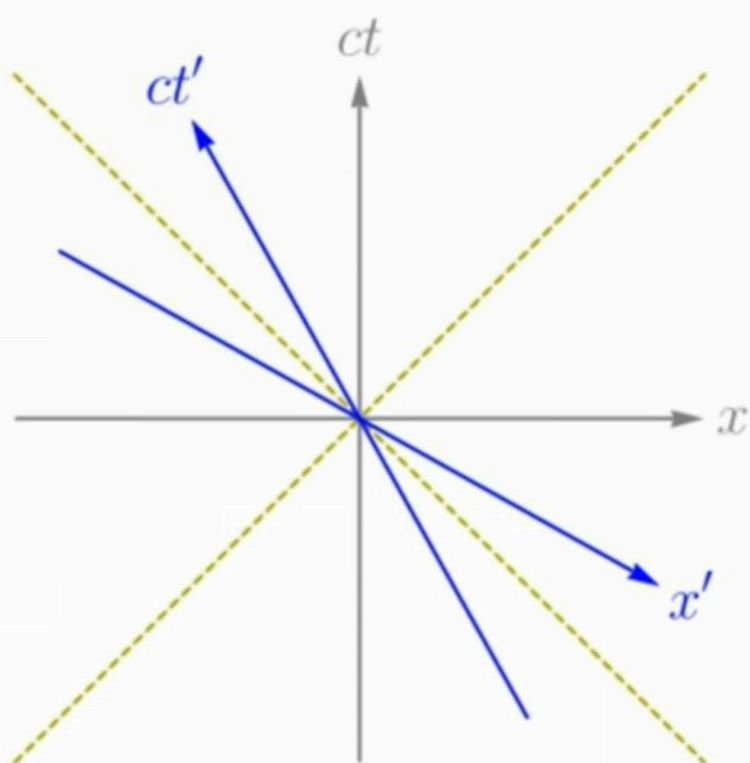
$$\int_{t_1}^{t_2} \delta L = \int_{t_1}^{t_2} \frac{d}{dt} \sum m_i \dot{x}_i \delta x_i = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta L = 0$$

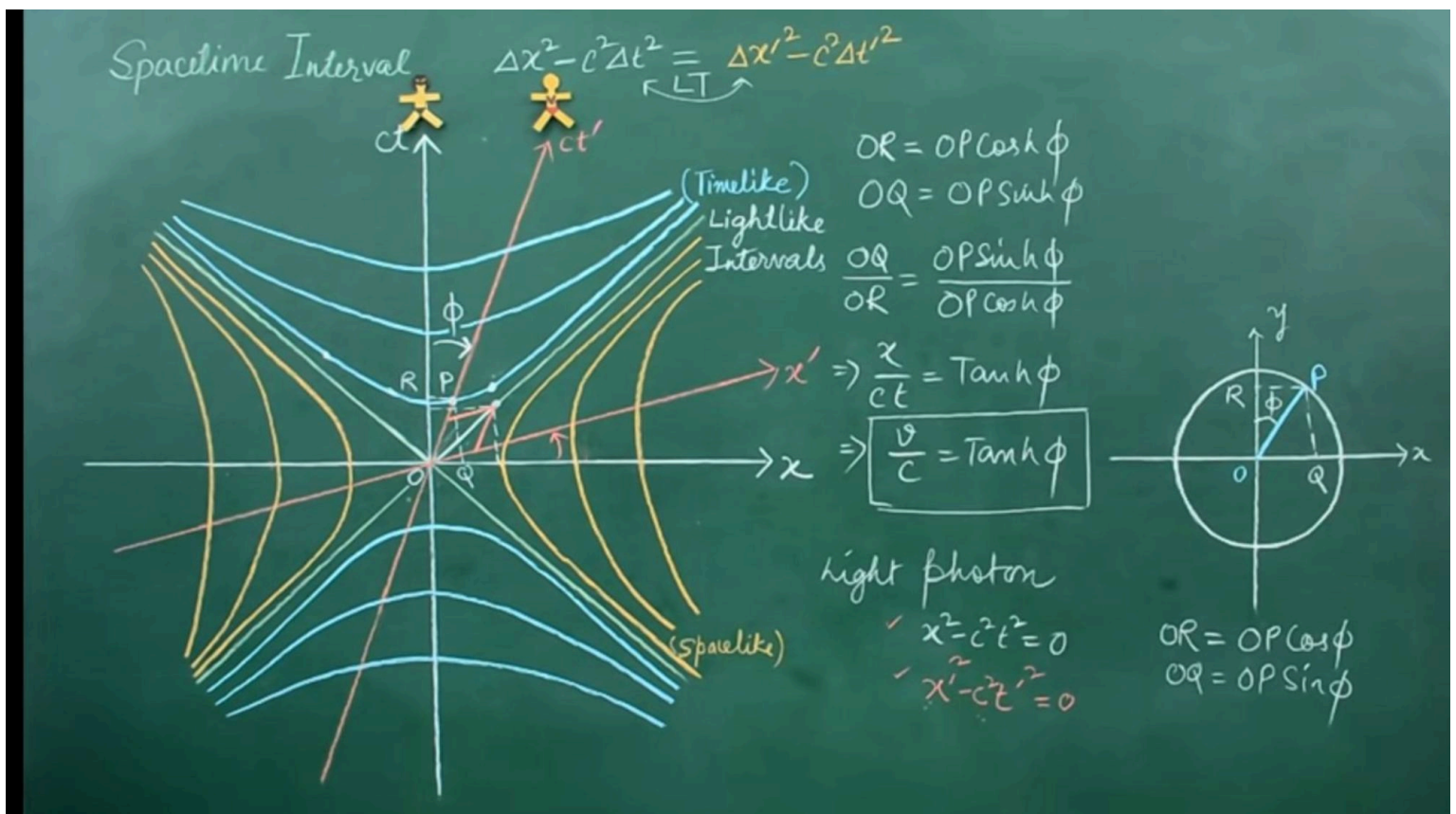
Minkowski Diagrams



$$v = -0.56c$$



Intervals & relation



Spacetime Interval

$$\Delta x^2 - c^2 \Delta t^2 \stackrel{LT}{=} \Delta x'^2 - c^2 \Delta t'^2$$



(Timelike)
Lightlike
Intervals

$$\begin{aligned} OR &= OP \cosh \phi \\ OQ &= OP \sinh \phi \\ \frac{OQ}{OR} &= \frac{OP \sinh \phi}{OP \cosh \phi} \end{aligned}$$

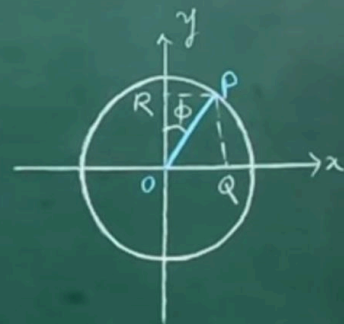
$$\frac{x}{ct} = \tanh \phi$$

$$\frac{v}{c} = \tanh \phi$$

light photon

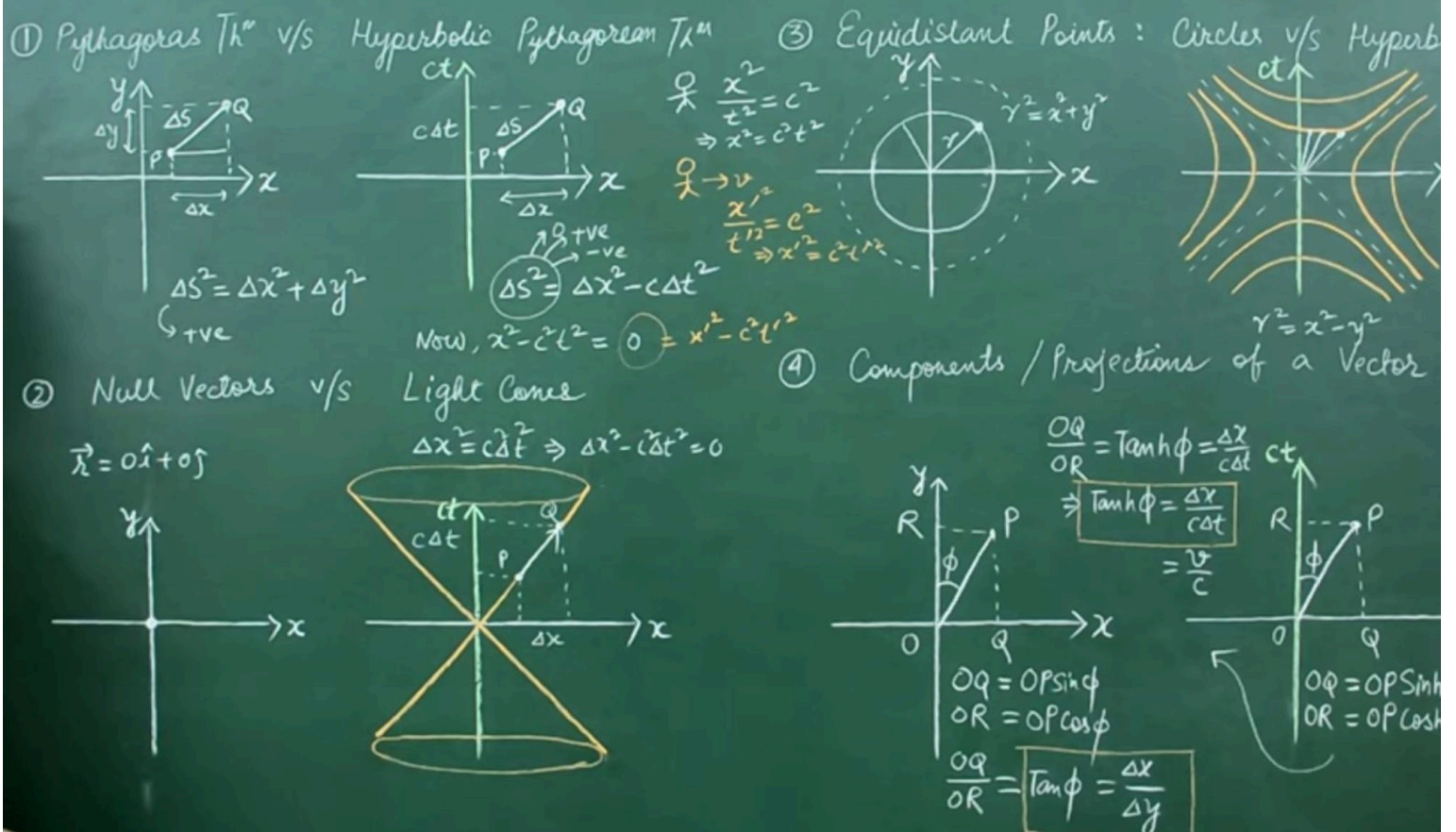
$$x^2 - c^2 t^2 = 0$$

$$x'^2 - c^2 t'^2 = 0$$

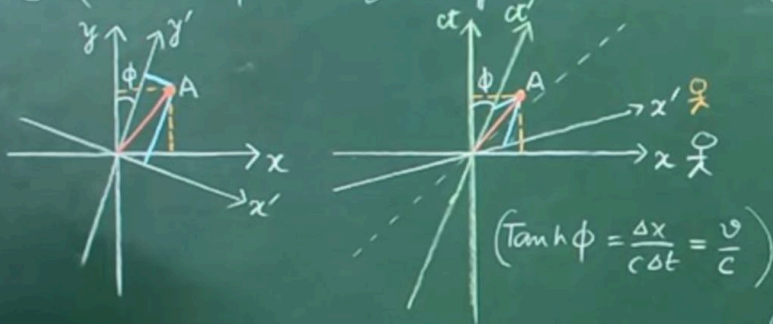


$$\begin{aligned} OR &= OP \cos \phi \\ OQ &= OP \sin \phi \end{aligned}$$

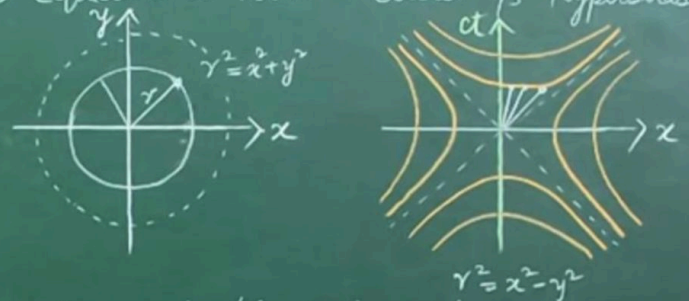
Cartesian vs Minkowski



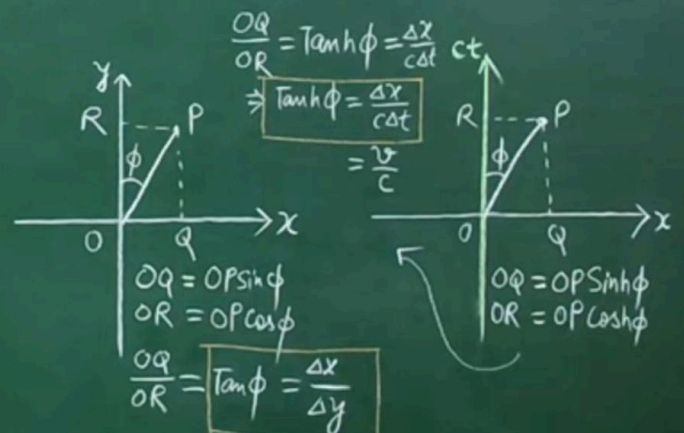
⑤ Rotations v/s Lorentz Transformations



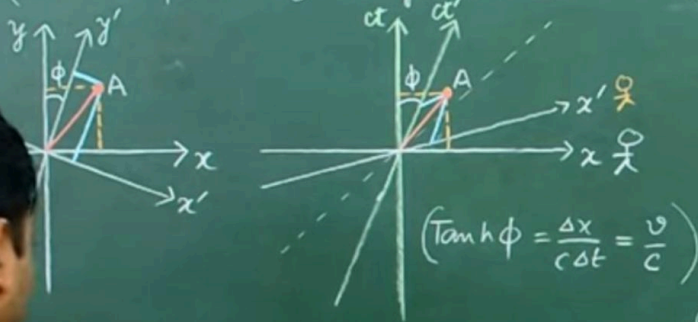
③ Equidistant Points: Circles v/s Hyperbolas



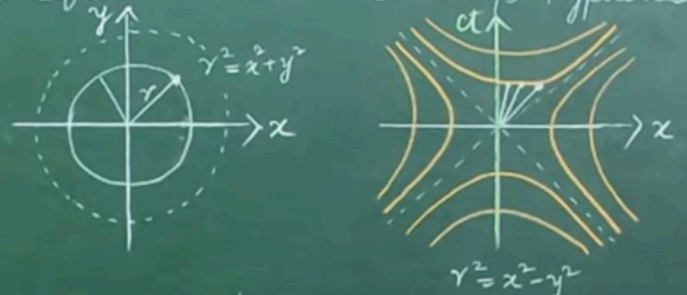
④ Components / Projections of a Vector



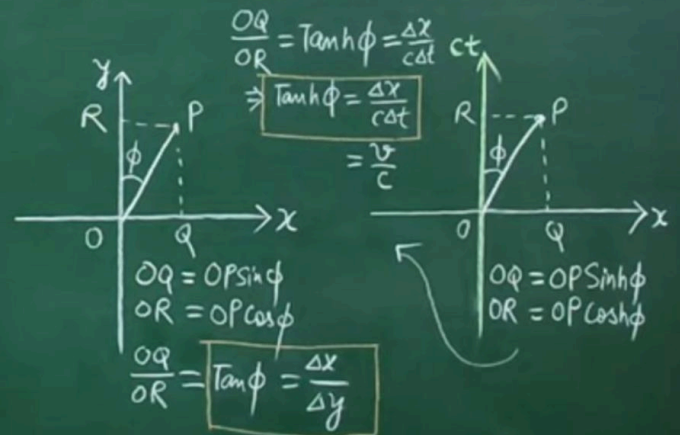
⑤ Rotations v/s Lorentz Transformations



③ Equidistant Points: Circles v/s Hyperbolas



④ Components / Projections of a Vector



ation :-

$$OA: \vec{r} = \vec{r}'$$

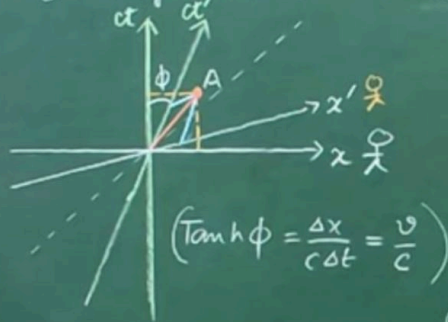
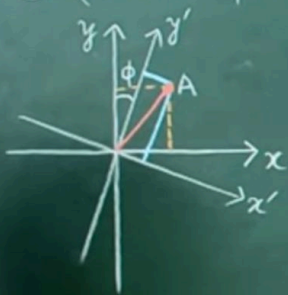
$$\Rightarrow x \hat{i} + y \hat{j} = x' \hat{i}' + y' \hat{j}'$$

$$x \hat{i} \cdot \hat{i}' + y \hat{j} \cdot \hat{i}' = x' \hat{i}' \cdot \hat{i}' + y' \hat{j}' \cdot \hat{i}'$$

$$\Rightarrow x \cos \theta - y \sin \theta = x'$$

$$\text{and } x \sin \theta + y \cos \theta = y'$$

⑤ Rotations v/s Lorentz Transformations



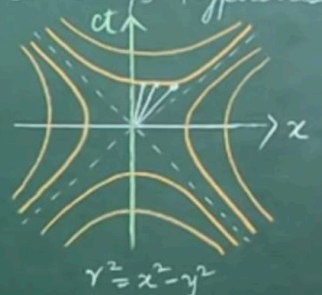
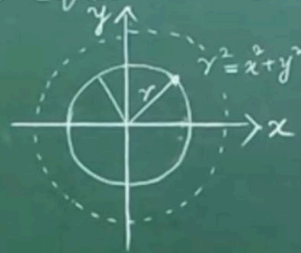
Rotation :-

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

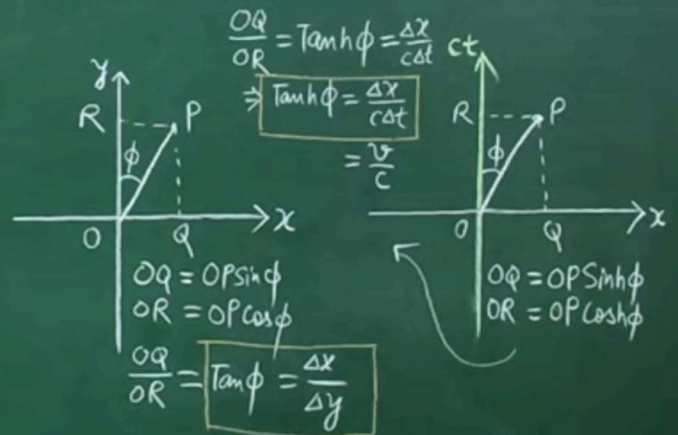
Rotation Matrix Operator

$$\Rightarrow \begin{cases} x \cos \theta - y \sin \theta = x' \\ x \sin \theta + y \cos \theta = y' \end{cases}$$

③ Equidistant Points : Circles v/s Hyperbolas

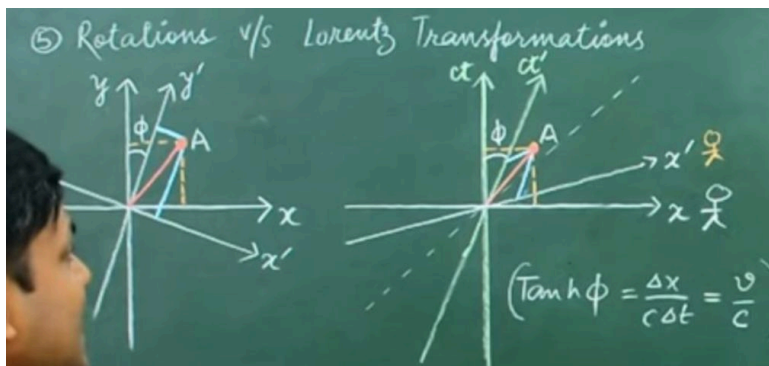


④ Components / Projections of a Vector



Transformation Matrix

⑤ Rotations v/s Lorentz Transformations



Notation :-

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

→ Rotation Matrix Operator

$$x \cos \theta + y \sin \theta = x'$$

Lorentz Transformations

$$\begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2} x \right) \\ ct' &= \gamma \left(ct - \frac{v}{c} x \right) \\ x' &= \gamma \left(x - \frac{v}{c} ct \right) \\ y' &= y \\ z' &= z \end{aligned}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (\tanh \phi)^2}} = \frac{1}{\sqrt{\text{sech}^2 \phi}} = \frac{1}{\text{sech} \phi} = \cosh \phi$$

Now,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c} \gamma \\ -\frac{v}{c} \gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh \phi & -\tanh \phi \cosh \phi \\ -\tanh \phi \cosh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

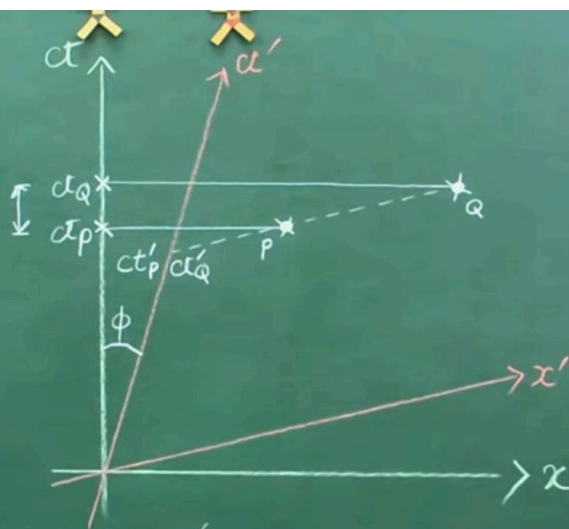
$$\Rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

→ Lorentz Transformation Matrix

Principle of non simultenity

Batman :- S-frame
Superman :- S'-frame
(v)

$$\boxed{\tanh \phi = \frac{v}{c}}$$



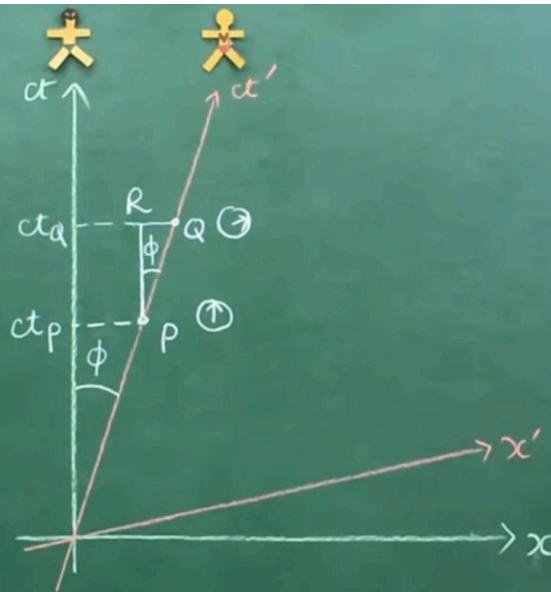
$ct'_P = ct'_Q$
 $\Rightarrow ct'_P - ct'_Q = 0$
 Physical events
 P and Q are
 simultaneous
 wrt S'

$$ct_P - ct_Q \neq 0$$

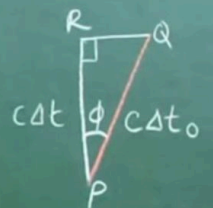
$\Rightarrow$ P and Q are NOT simultaneous
 wrt S-frame.

Time dilation

$$\begin{aligned} \tanh \phi &= v/c \\ \Rightarrow \tanh^2 \phi &= \frac{v^2}{c^2} \\ \Rightarrow 1 - \tanh^2 \phi &= 1 - \frac{v^2}{c^2} \\ \Rightarrow \operatorname{sech}^2 \phi &= 1 - \frac{v^2}{c^2} \\ \Rightarrow \operatorname{sech} \phi &= \sqrt{1 - v^2/c^2} \\ \Rightarrow \cosh \phi &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ PQ &= (ct'_Q - ct'_P) \\ &= c(t'_Q - t'_P) \\ &= c\Delta t_0 \end{aligned}$$



Acc to observer in S-frame

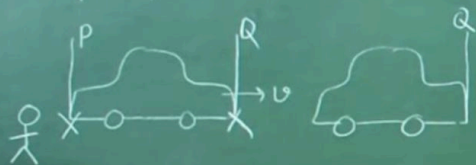


$$\begin{aligned} PR &= ct_Q - ct_P \\ &= c(t_Q - t_P) \\ &= c\Delta t \end{aligned}$$

$$\begin{aligned} PR &= PQ \cosh \phi \\ \Rightarrow \cancel{c}\Delta t &= \cancel{c}\Delta t_0 / \sqrt{1 - v^2/c^2} \Rightarrow \Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

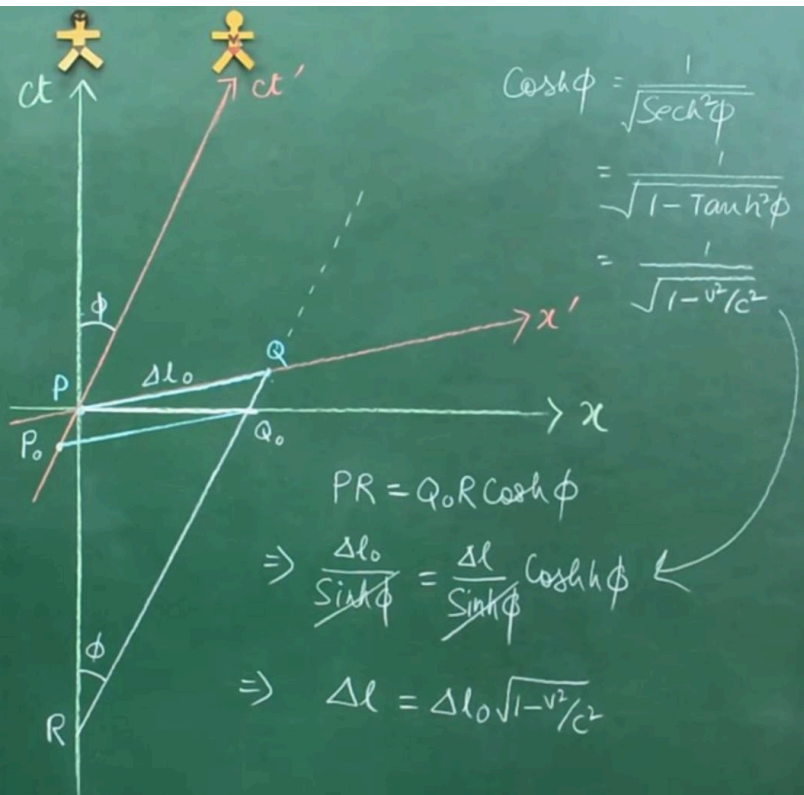
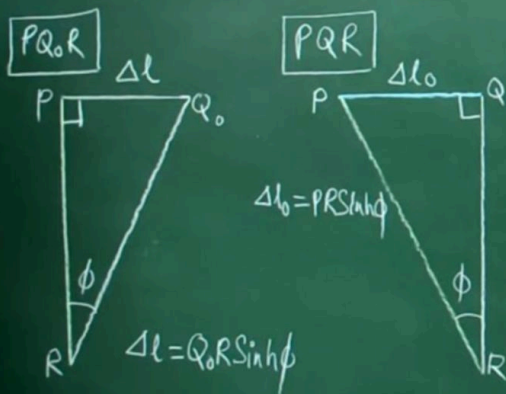
Length Contraction

$$\tanh \phi = v/c$$



Wrt S' -frame: $-PQ = \Delta l_0$

Wrt S -frame: $-PQ_0 = \Delta l$



Relativistic Velocity Addition

$$\tanh \phi = v/c$$

Speed of S' wrt S :- v

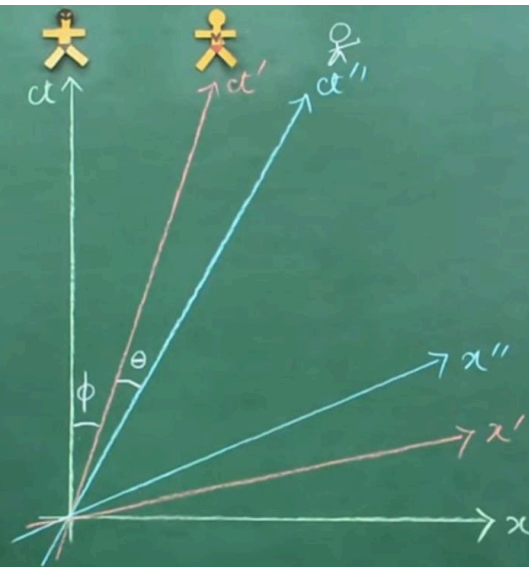
Speed of S'' wrt S' :- u'

Speed of S'' wrt S :- $u = ?$

$$\tanh(\theta + \phi) = \frac{u}{c}$$

$$\Rightarrow \frac{\tanh \theta + \tanh \phi}{1 + \tanh \theta \tanh \phi} = \frac{u}{c}$$

$$\Rightarrow \frac{\frac{u'}{c} + \frac{v}{c}}{1 + \frac{u'v}{c^2}} = \frac{u}{c} \Rightarrow u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$



Integrating it further, we get

$$x = -\frac{1}{2}gt^2 + C'$$

Initially at $t=0$ the distance of the stone from the earth $x=h$. This gives $h=C'$, and hence

$$x = -\frac{1}{2}gt^2 + h$$

Finally, when at $t=T$, the stone touches the ground $x=0$. Therefore

$$h - \frac{1}{2}gT^2 = 0 \quad \text{or} \quad T = \sqrt{2h/g} \quad \dots(23)$$

Now,
$$\frac{d^2y}{dt^2} = -2\omega \frac{dx}{dt} \cos \phi \quad \text{or} \quad \frac{d^2y}{dt^2} = 2\omega gt \cos \phi$$

Integrating it, we get

$$\frac{dy}{dt} = \omega gt^2 \cos \phi \quad \text{and hence} \quad y = \frac{\omega gt^3}{3} \cos \phi \quad \dots(24)$$

The constants of integration in (24) have been taken zero, because initially the stone has no displacement and velocity in the Y -direction is zero.

Finally, at $t=T$, the maximum horizontal displacement will be given by

$$Y = \frac{\omega g T^3}{3} \cos \phi \quad \dots(25)$$

Substituting the value of T from eq (23), we get

$$Y = \frac{\omega g}{3} \left[\frac{2h}{g} \right]^{3/2} \cos \phi \quad \text{or} \quad Y = \left[\frac{8}{9g} \right]^{1/2} h^{3/2} \omega \cos \phi \quad \dots(26)$$

Thus a freely falling body at latitude ϕ is displaced horizontally due east by Coriolis force by an amount given by eq. (26). At the equator the easterly deflection is obtained to be

$$Y = \left[\frac{8}{9g} \right]^{1/2} h^{3/2} \omega \quad \dots(27)$$

11.6. FOUCAULT'S PENDULUM

Experimentally the fact, that the earth rotates about its axis and the frame attached to it is a noninertial frame, is demonstrated by the Foucault's pendulum. The related experiment was first performed by Foucault in 1851.

In Foucault's pendulum, a very heavy bob is suspended by means of a long strong wire similar to a simple pendulum. Thus if it is once vibrated, it vibrates for considerably large time with a large period (≈ 17 sec). Foucault took in his experiment a bob of 28 kg mass and a wire of 70 metres long. The upper end of the wire is attached to a rigid support in such a way that the pendulum may vibrate with equal freedom in any direction.

When Foucault's pendulum is oscillated in the northern hemisphere of the earth, its plane of oscillation rotates from east to west.

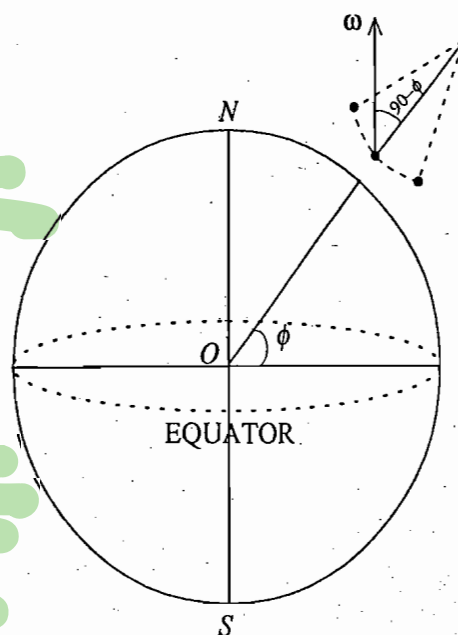


Fig. 11.7 : Foucault's pendulum

(or clockwise as seen from above). It is not possible that the point of support might have turned the plane of vibration. The only possibility is this that the floor, i.e., the earth under the pendulum is rotating. If such a pendulum is imagined to vibrate at the north pole of the earth, its plane of oscillation will remain fixed in an inertial frame or solar frame of reference. However, the earth under the pendulum is rotating once every 24 hours, therefore an observer on the earth will see that the plane of oscillation is turning from east to west, opposite to the earth's rotation. At any other latitude ϕ , the angular velocity ω can be resolved into two components : vertical component $\omega \sin \phi$ and horizontal component $\omega \cos \phi$ in the north-south direction. Obviously the later component will not have any perceptible effect on the pendulum. The vertical component will make the plane of oscillation to rotate with an angular velocity $\omega \sin \phi$. Thus the time of one rotation of the plane of oscillation will be given by

$$T = \frac{2\pi}{\omega \sin \phi} \quad \dots(28)$$

The earth rotates once in 24 hours with an angular velocity ω , hence

$$24 \text{ hours} = \frac{2\pi}{\omega} \quad \text{or,} \quad T = \frac{24 \text{ hours}}{\sin \phi} \quad \dots(29)$$

The rotation of the plane of oscillation will be clockwise in the northern hemisphere and opposite to it in the southern hemisphere. Actually, this rotation, as seen by the observer on the earth, is due to Coriolis force. When the pendulum moves along a horizontal line, the Coriolis force acts in the perpendicular direction to its velocity and hence the plane of oscillation turns. Thus, *the rotation of the plane of oscillation of Foucault's pendulum demonstrates the fact that the earth rotates about its axis.*

Ex. 1. A bullet is fired horizontally in the north direction with a velocity of 500 m/sec. at 30° N latitude. Calculate the horizontal component of Coriolis acceleration and the consequent deflection of the bullet as it hits a target 250 metres away. Also determine the vertical displacement of the bullet due to gravity. If the mass of the bullet is 10 gm. find the Coriolis force. (Ajmer 1988)

Solution : If X-axis is taken vertically, Z-axis towards north and Y-axis along east, then the velocity of the bullet is $v = 500$ km/sec. and angular velocity $\omega = \omega (\hat{k} \cos 30^\circ + \hat{i} \sin 30^\circ)$, because the angular velocity vector ω of the earth is directed parallel to its axis and is inclined at 30° to the horizontal.

$$\text{Here, } \omega = \frac{2\pi}{24 \times 60 \times 60} = 7.2 \times 10^{-5} \text{ rad./sec.}$$

Hence Coriolis acceleration

$$\begin{aligned} &= 2 \omega \times v = 2\omega (\hat{k} \cos 30^\circ + \hat{i} \sin 30^\circ) \times 500 \hat{k} \\ &= -2 \times 7.2 \times 10^{-5} \times 500 \times \frac{1}{2} \hat{j} = 0.036 \text{ m/sec}^2 \text{ towards west.} \end{aligned}$$

$$\text{Time of journey, } t = \frac{250}{500} = \frac{1}{2} \text{ sec.}$$

Deflection of the bullet due to the Coriolis acceleration

$$= \frac{1}{2} at^2 = \frac{1}{2} \times 0.036 \times \left(\frac{1}{2}\right)^2 = 4.5 \times 10^{-3} \text{ m.}$$

Vertical displacement of the bullet due to the gravity

$$= \frac{1}{2} gt^2 = \frac{1}{2} \times 9.8 \times \left(\frac{1}{2}\right)^2 = 1.23 \text{ m.}$$

Coriolis force

$$= -2m \omega \times v$$

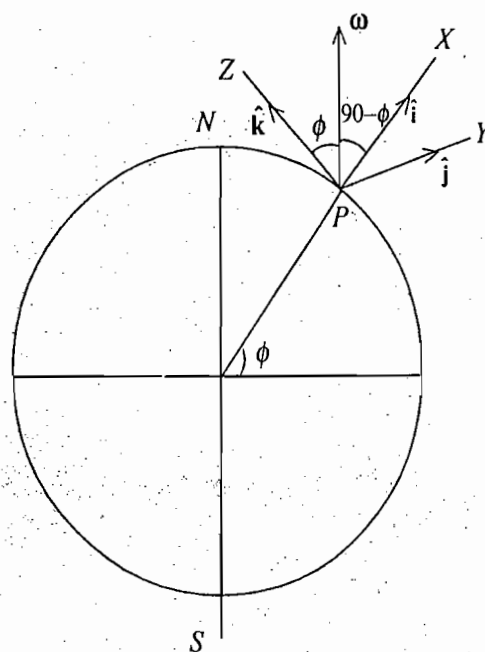


Fig. 11.8

Since $\sin^2 \theta = 1 - \cos^2 \theta = (1 - \cos \theta)(1 + \cos \theta)$,

$$V(\theta) = \frac{I_3^2 \omega_3^2}{2I_1} \left[\frac{1 - \cos \theta}{1 + \cos \theta} + \lambda \cos \theta \right]$$

For minima in V - θ curve,

$$\frac{dV}{d\theta} = \frac{I_3^2 \omega_3^2}{2I_1} \left[\frac{\sin \theta}{1 + \cos \theta} + \frac{(1 - \cos \theta) \sin \theta}{(1 + \cos \theta)^2} - \lambda \sin \theta \right] = 0$$

$$\text{or} \quad \sin \theta \left[\frac{1}{1 + \cos \theta} + \frac{(1 - \cos \theta)}{(1 + \cos \theta)^2} - \lambda \sin \theta \right] = 0 \quad \dots(147)$$

which gives $\sin \theta = 0$ or $\theta = 0$ i.e., there can be minima at $\theta = 0$. Further for minima at $\theta = 0$,

$$\left(\frac{d^2 V}{d\theta^2} \right)_{\theta=0} > 0 \quad \text{or} \quad \left[\cos \theta \left\{ \frac{1}{1 + \cos \theta} + \frac{(1 - \cos \theta)}{(1 + \cos \theta)^2} - \lambda \right\} \right]_{\theta=0} > 0$$

$$\text{or} \quad \frac{1}{2} - \lambda > 0 \quad \text{or} \quad \frac{1}{2} > \lambda \quad \dots(148)$$

Thus at $\theta = 0$, (or top with its symmetry axis vertical), the spinning motion is stable for

$$\frac{1}{2} > \lambda \quad \text{or} \quad \frac{1}{2} > \frac{2I_1 Mgl}{I_3 \omega_3^2}$$

which is valid for large ω_3 or rapidly spinning top. In other words, for vertical spinning top,

$$\omega_3^2 > 4I_1 Mgl / I_3 \quad \dots(149)$$

The minimum spin angular frequency of the top below which the top cannot spin stably about vertical axis is when $\omega_3 = \omega_{\min}$, given by

$$\omega_{\min}^2 = \frac{4I_1 Mgl}{I_3} \quad \text{or} \quad \omega_{\min} = \left[\frac{4I_1 Mgl}{I_3} \right]^{1/2} \quad \dots(150)$$

Thus if initially $\omega_3 > \omega_{\min}$, a top with its axis vertical will spin continuously and this is why it is called a **sleeping top**. This $\omega_{\min}$ is the critical angular velocity, above which vertical motion of the top is possible. The expression (150) is identical with eq. (119) for the minimum frequency for uniform precession with $\theta_0 = 0$. In practice, friction gradually reduces the spin frequency (ω_3) and when it is below the critical angular velocity ($\omega_{\min}$), then precession combined with nutation will be introduced. Further reduction of energy will finally cause the top fall down.

10.17. GYROSCOPE

A gyroscope is a heavy symmetrical body (top) in the form of a heavy disc or flywheel, rotating at a very high speed about its axle (figure axis). It is mounted in gimbals so that the disc and axle are both free to turn as a whole about one or more special axes, keeping the centre of gravity of the moving system to be stationary.

A gyroscope is illustrated in fig. 10.21. A disc D , having its axle AB mounted in gimbals, is given a spin angular velocity ω . If the outer gimbal is turned through an angle, the spin axis of the disc will tend to point in the same direction as previously [fig. 10.21 (b)]. This is in accordance with the principle of conservation of angular momentum ($J = I\omega = \text{constant}$). If we move the gyroscope around the room, we will see that AB

always points in the same direction. We have assumed of course that the friction in gimbal bearings is negligible. Let the gyro-axis AB be horizontal and parallel to east west direction. We shall observe that the axis AB gradually inclines itself relative to the horizontal and after six hours it becomes vertical. This apparent rotation of the gyro-axis AB is due to the rotation of the earth. As the apparatus moves, the gyro-axis AB remains fixed in space.

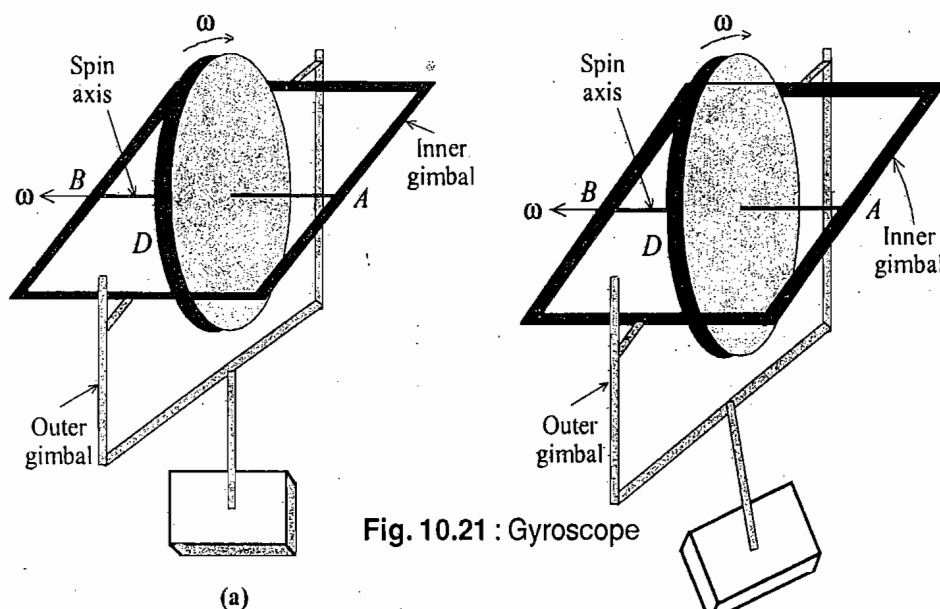


Fig. 10.21 : Gyroscope

In general, the direction of the spin axis of the gyroscope remains fixed even when the outer gimbal, which is attached to some object, moves freely in space. Due to this property, a gyroscope finds many applications in devices where we want to maintain the direction. Gyro-compass, working on this principle, is used in ships, aeroplanes and submarines to find direction. Other practical applications are found in missiles, satellites or other moving vehicles.

Effect of external torque on gyroscope : If external torque is suitably applied on the gyroscope, it will undergo the precession and nutation motions, as described earlier. For rapid motion of the disc, the condition of the fast top is satisfied so that the amplitude of nutation is always very small and is damped out by the method of mounting (friction). Then only precession is observed in the gyroscope and in such a case the mathematical treatment becomes relatively much simpler. This generalization can be done by considering the case of heavy symmetrical top.

If l is the position vector along the figure axis from the fixed point O to the centre of gravity [Fig. 10.22], then the gravitational torque τ acting on the top is

$$\tau = l \times Mg$$

...(151)

where g is the acceleration due to gravity in the downward direction and the magnitude of the torque is

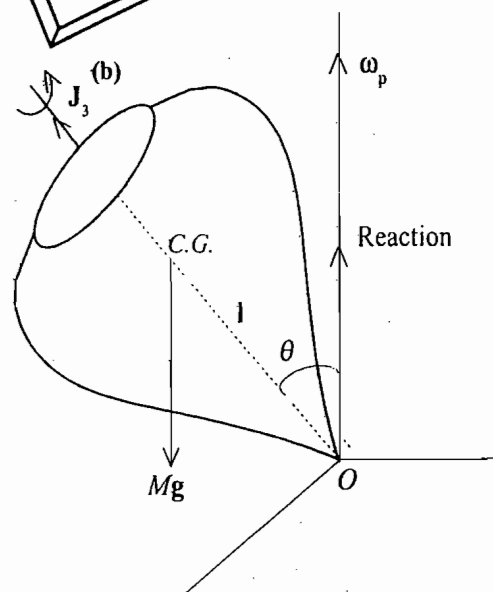


Fig. 10.22 : Precession under gravitational torque.

$Mgl \sin \theta$. If J_3 is the angular momentum vector along the figure axis ($J_3 = I_3 \omega_3$) and ω_p , the precession vector, along the vertical, then the sense and magnitude of precession is given by

$$\omega_p \times J_3 = \tau \quad \dots(152 a)$$

$$\text{or} \quad \omega_p \times J_3 = l \times Mg$$

$$\text{or} \quad \omega_p J_3 \sin \theta = l Mg \sin \theta$$

$$\text{or} \quad \omega_p = \frac{Mgl}{J_3} = \frac{Mgl}{I_3 \omega_3} \quad \dots(152 b)$$

which is the same expression, as deduced earlier for the mean precession frequency. In the present discussion, τ is perpendicular to J_3 , hence J_3 will remain constant in magnitude similar to speed remains constant in circular motion under the action of centripetal force. From (152) we see that the precession rate is proportional to the torque. However, in case of a non-spinning body, it is the angular acceleration which is proportional to the torque.

Now any torque about the fixed point or centre of mass can be put in the form $\mathbf{r} \times \mathbf{F}$ similar to eq. (151), the resulting precession rate for a fast spinning gyroscope can always be derived from eq. (152). In almost all the technological applications of the gyroscope, its equilibrium behaviour can be deduced from eq. (152), i.e.,

$$\omega_p \times J_3 = \mathbf{r} \times \mathbf{F} \quad \dots(153)$$

If a gyroscope is free from external torques, the spin axis of the gyroscope will always maintain its original direction relative to an inertial frame and we have described this behaviour in the beginning of this article. As shown by eq. (152), through the precession phenomena, the gyroscope can sense the applied torques, because in absence of the external torque, a gyroscope stops precessing.

Ex. 1. Two point masses, each of mass m , are connected by a massless rigid rod of length $2a$, forming a dumbbell. This dumbbell is rotating with constant angular velocity ω about an axis which makes an angle θ with the rod. Find the magnitudes and the directions of the angular momentum and the torque, applied to the system. How are the results affected, if an identical dumbbell is symmetrically fixed with the first dumbbell and the system is rotated with an angular velocity ω about the same axis.

Solution : Let the dumbbell [Fig. 10.23] rotate with an angular velocity ω about the axis AOB in an inertial coordinate system. The angular momentum $\mathbf{J}$ of the two masses is

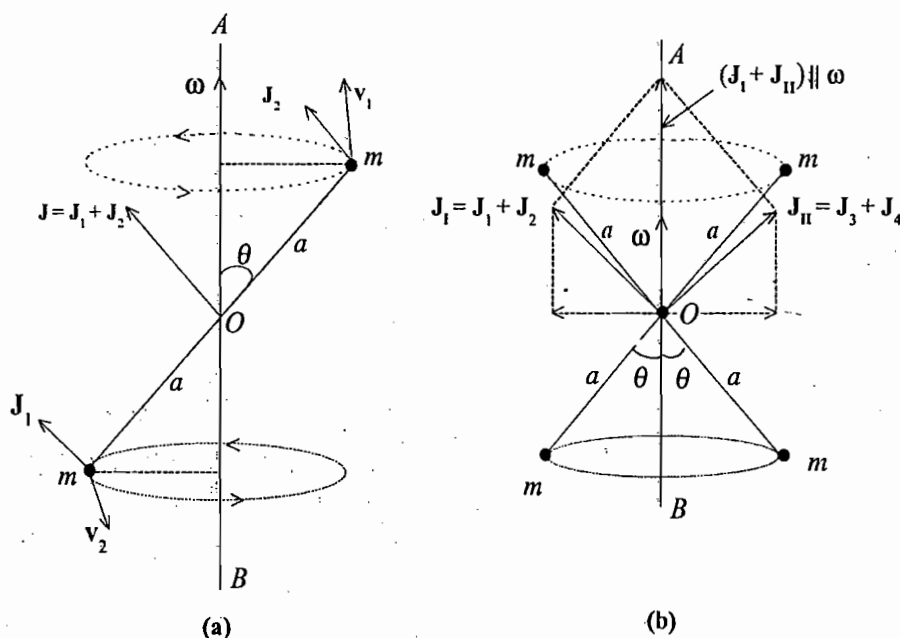


Fig. 10.23

Special Theory of Relativity- Lorentz Transformations

12.1. INTRODUCTION

In Chapter 1, we discussed the concept of space and time and gave the idea of inertial frames. An inertial frame is an unaccelerated coordinate system. In such a frame, if force is not acting on a particle, it will remain either at rest or will move with the constant velocity, *i.e.*, the law of inertia (Newton's first law) holds true. We have also shown that a frame, moving with constant velocity relative to an inertial frame, is also inertial one.

In Newtonian mechanics, space and time are completely separable and the transformations connecting the space-time coordinates of a particle are the Galilean transformations. These transformations are valid as far as Newton's laws are concerned, but fail in the field of electrodynamics. Principle of relativity, when applied to the electromagnetic phenomena, asserts that the speed of light in vacuum is a constant of nature. This statement has been confirmed by several experiments and led Einstein to formulate the special theory of relativity. In view of this theory, space and time are not independent of each other and the correct transformation equations are Lorentz transformations. In this chapter, first we discuss the Galilean transformations and their failures and then we deduce the Lorentz transformations. Finally, we discuss the consequences of Lorentz transformations, namely length contraction, time dilation, simultaneity, velocity addition etc.

12.2. GALILEAN TRANSFORMATIONS

At any instant, the coordinates of a point or particle in space will be different in different coordinate systems. The equations which provide the relationship between the coordinates of two reference systems are called *transformation equations*.

In Chapter 1, we have shown that a frame S' which is moving with constant velocity v relative to an inertial frame S , is itself inertial.

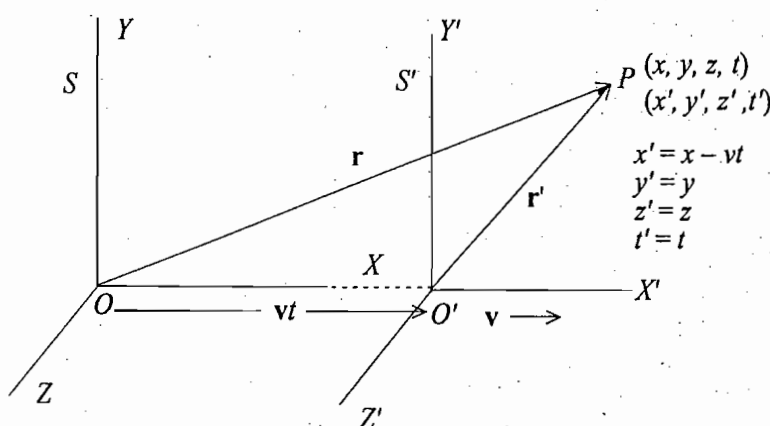


Fig. 12.1 : Representation of Galilean transformations

For convenience, if we assume (i) that the origins of the two frames coincide at $t = 0$, (ii) that the coordinate axes of the second frame are parallel to that of the first and (iii) that the velocity of the second frame relative to the first is v along X -axis, then the position vectors of a particle at any instant t in the two frames are related by the equation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \quad \dots(1)$$

In the component form, the coordinates are related by the equations

$$x' = x - vt ; \quad y' = y ; \quad z' = z \quad \dots(2)$$

Eq. (1) or (2) expresses the transformation of coordinates from one inertial frame to another and they are referred as **Galilean transformations**.

The form of eq. (1) or (2) depends, of course, on the relative motion of two frames of reference, but it also depends upon certain assumptions regarding the nature of time and space. It is assumed that the time t is independent of any particular frame of reference i.e., if t and t' be the times recorded by the observers O and O' of an event occurring at P , then $t' = t$. If we add the equation $t' = t$, then the **Galilean transformation equations** are expressed as

$$x' = x - vt ; \quad y' = y ; \quad z' = z ; \quad t' = t \quad \dots(3)$$

We can also consider that frame S is moving with velocity $-v$ along the negative X -axis with respect to S' frame. Then the transformation equations from frame S' to S are

$$x = x' + vt', \quad y = y' \quad z = z' \quad t = t' \quad \dots(3')$$

These are known as **inverse Galilean transformations**.

The other assumption, regarding the nature of the space, is that the distance between two points (or two particles) is independent of any particular frame of reference. Evidently, this assumption is expressed by the form of the transformation eq. (1) or (3). If a rod has length L in the frame S with the end coordinates x_1 and x_2 , then $L = x_2 - x_1$.

If at the same time the end coordinates of the rod in S' are x_1' and x_2' , then $L' = x_2' - x_1'$. But for any time t , from eq. (3), we have

$$x_2' - x_1' = x_2 - x_1$$

$$\text{Therefore,} \quad L' = L \quad \dots(4)$$

Thus the length or distance between two points is invariant under Galilean transformations.

Differentiating eq.(1) with respect to time, we get

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} + \frac{d\mathbf{r}'}{dt} = \mathbf{v} + \frac{d\mathbf{r}'}{dt'} \quad [\because t = t']$$

$$\text{or} \quad \mathbf{u} = \mathbf{v} + \mathbf{u}' \quad \dots(5)$$

where $\mathbf{u}$ and $\mathbf{u}'$ are the observed velocities in S and S' frames respectively.

Eq. (5) transforms the velocity of a particle from one frame to another and is known as **Galilean (or classical) law of addition of velocities**.

Again differentiating eq. (5) with respect to time t , we have

$$\frac{d\mathbf{u}}{dt} = 0 + \frac{d\mathbf{u}'}{dt} = \frac{d\mathbf{u}'}{dt'} \quad [\because t = t']$$

$$\text{or} \quad \mathbf{a} = \mathbf{a}' \quad \dots(6)$$

Hence according to Galilean transformations, the accelerations of a particle relative to S and S' frames are equal.

It is to be mentioned that the Galilean transformations are based basically on two assumptions :

(1) *There exists a universal time t which is the same in all reference systems.*

(2) *The distance between two points in various inertial systems is the same.*

Thus if any two events occur simultaneously for any observer, then they must occur simultaneously for all observers. In other words, *the time interval between two given events must be identical for all systems of reference.*

12.3. PRINCIPLE OF RELATIVITY

Absolute velocity of a body has no meaning. The velocity has a meaning only when it is measured relative to some other body or frame of reference. If two bodies are moving with uniform relative velocity, it is impossible to decide which of them is at rest or which of them is moving. This is known as *principle of relativity*. However, acceleration has an absolute meaning. For example, if we are sitting in a windowless accelerated aircraft, we can perform an experiment and measure its acceleration. But if the aircraft is moving with uniform velocity, we cannot measure its velocity. Of course, we measure its velocity relative to a body outside. Thus the principle of relativity can be alternatively stated as follows :

It is impossible to perform an experiment which will measure the state of uniform velocity of a system by observations, confined to that system.

The motion of a body itself has no meaning unless, we do not know with respect to which this motion has been measured. This led Newton to think about the absolute space and it represents an absolute frame with respect to which every motion should be measured. However, in view of this principle of relativity, we can not perform an experiment which will measure the uniform velocity of a reference system relative to the absolute frame by observations confined to that system.

In the unaccelerated windowless ship all experiments performed inside it will appear the same whether this ship is stationary or in uniform motion. Newton stated the principle of relativity as follows :

The motions of bodies included in a given space are the same among themselves whether that space is at rest or moving uniformly forward in a straight line.

Study of the physical laws involves the measurements of accelerations, forces etc among bodies. The principle of relativity can be stated in an elegant form as follows :

The basic laws of physics are identical in all inertial systems which move with uniform velocity with respect to one another.

This principle is called *Galilean or Newtonian principle of relativity* and sometimes it is named as *hypothesis of Galilean invariance*. In fact, the principle of relativity is a fundamental postulate and is entirely consistent with the theory of special relativity. If any two inertial systems, moving with constant relative velocity, are connected by Galilean transformations, the principle of relativity is modified as :

The basic laws of physics are invariant in form in two reference systems connected by Galilean transformations.

This statement is somewhat special than the principle of relativity in the sense that it means the assumptions that the time and the space intervals are independent of the frame of reference. We shall see later in the theory of special relativity that the Galilean transformations are not correct, but the appropriate exact transformation equations are the Lorentz transformation equations for connecting any two frames in uniform relative motion. Thus, the principle of relativity may be stated as :

The basic laws of physics are invariant in form in two inertial frames connected by Lorentz transformations.

12.4. TRANSFORMATION OF FORCE FROM ONE INERTIAL SYSTEM TO ANOTHER

Suppose that the force F on a particle of mass m in the frame S is represented by Newton's second law

